BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY

A HISTORICAL AND CRITICAL REVIEW OF MATHEMATICAL SCIENCE

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BULLETIN
OF THE
AMERICAN MATHEMATICAL SOCIETY

THE INTERNATIONAL CONGRESS AT TORONTO*

The seventh International Mathematical Congress was opened on Monday morning, August 11, 1924, in the Convocation Hall of the University of Toronto by Professor Ch. de la Vallée Poussin, President of the International Mathematical Union. Addresses of welcome were delivered by the Honorable H. S. Beland on behalf of the Government of the Dominion, and by Sir Robert Falconer, President of the University of Toronto. At this session, Professor J. C. Fields was elected President of the Congress; Vice-Presidents were appointed for the sessions of the Congress, which were to be held in eight sections as follows:

SECTION I. Algebra, Theory of Numbers, Analysis.
SECTION II. Geometry.
SECTION III. (a) Mechanics, Mathematical Physics.
SECTION III. (b) Astronomy, Geophysics.
SECTION IV. (a) Electrical, Mechanical, Civil and Mining Engineering.
SECTION IV. (b) Aeronautics, Naval Architecture, Ballistics, Radiotelegraphy.
SECTION V. Statistics, Actuarial Science, Economics.
SECTION VI. History, Philosophy, Didactics.

* A report prepared for this BULLETIN by Arnold Dresden.

These Sections began their meetings on Monday afternoon, and continued every morning throughout the week, with the exception of Thursday which was set aside for an all day excursion to Niagara Falls.

The afternoons and evenings were devoted to general lectures, and to social functions. The meetings of the British Association for the Advancement of Science which had started during the preceding week overlapped with the earlier sessions of the Congress, thus furnishing an opportunity for hearing many of the most eminent scientists in many fields.

Not enough praise can be given to the Committee on Entertainment, to the officials of the Province of Ontario, of the City of Toronto, and of the University, and to the citizens of Toronto, who provided lavish opportunity for social intercourse in the form of garden parties, a conversazione in Hart House, an excursion around the City of Toronto, etc.

On Saturday afternoon the Congress paid homage to the students of the University of Toronto who laid down their lives in the war. A wreath was placed in the newly erected Memorial Tower of the University; short addresses were made by Professor de la Vallée Poussin and by President Sir Robert Falconer; and it was announced that arrangements had been made for inserting in the wall of the Tower a medallion bearing the following inscription: "To the heroes of the University of Toronto, the members of the International Mathematical Congress, Toronto, 1924."

A special convocation of the University of Toronto was held on Wednesday afternoon. Honorary degrees were conferred upon Sir David Bruce, Sir Charles Parsons, Sir John Russell, Sir Ernest Rutherford of the British Association for the advancement of Science, and upon Professors F. Severi, Ch. de la Vallée Poussin, V. Stekloff and G. Koenigs of the International Mathematical Congress.

The general assembly of the International Mathematical Union took place on Friday morning. Of the six delegates elected by the American Section of the Union, Professors

Coble, Richardson, and Snyder were present.* Professor S. Pincherle, of the University of Bologna, was elected President of the Union for the ensuing quadrennium; Professors G. Koenigs and A. Demoulin were reelected as Secretary and Treasurer respectively. Professors G. A. Bliss, E. Phragmen, and H. Fehr were elected as Vice-Presidents in place of Professors Dickson, Larmor, and Volterra, whose terms expired, while Professors P. Appell and W. H. Young continue as Vice-Presidents. As Honorary Presidents of the Union were elected Professors L. E. Dickson, J. C. Fields, H. Lamb, G. Mittag Leffler, Emile Picard, Ch. de la Vallée Poussin, and V. Volterra.

A Committee on International Bibliography was appointed, consisting of Professors Archibald, Bortolotti, Fréchet, van der Woude, and W. H. Young.

The following announcement was made by the American Section of the Union and received the endorsement of the delegations from Italy, Holland, Sweden, Denmark, Norway, and England.

"The American Section of the International Mathematical Union has unanimously passed the following resolution and requests that it be transmitted to the International Research Council by the Executive Committee of the International Mathematical Union:

"Resolved that the American Section of the International Mathematical Union requests the International Research Council to consider whether the time is ripe for the removal of the restrictions on membership now imposed by the rules of the Council.

"The American Section also wishes to announce that it intends to present to the National Research Council of the United States resolutions requesting action in this direction."

The closing session of the Congress took place on Saturday afternoon.

Professor Synge, Secretary of the Congress, announced

* See this BULLETIN, vol. 30 (1924), p. 377.

that 387 persons, representing 25 countries, had registered as members of the Congress, the distribution being as follows: Argentina 2, Belgium 7, Canada 67, Cuba 1, Czecho Slovakia 3, Denmark 3, France 26, Georgia 1, Great Britain 39, Holland 4, India 2, Ireland 3, Italy 11, Mexico 1, Norway 5, Poland 3, Portugal 2, Rumania 1, Russia 5, Spain 3, Sweden 3, Switzerland 4, Ukrania 1, United States 189, Yugo Slavia 1.

A resolution was unanimously adopted stating that the Congress hears with pleasure that the Royal Irish Academy contemplates the publication of a collected edition of the works of Sir William Rowan Hamilton.

Resolutions were also adopted expressing to the government of the Dominion, of the Province of Ontario, and of the City of Toronto, and to the many persons and organizations who had contributed to the success of the meeting the recognition and thanks of the Congress.

The number of papers read at the meetings of the various Sections reached the enormous total of 235. In addition to these, the following 7 lectures were given before general sessions of the Congress:

C. Störmer, *Modern Norwegian Researches on the Aurora Borealis.*

F. Severi, *Géométrie algébrique.*

E. Cartan, *La théorie des groupes et les recherches récentes de géométrie différentielle.*

W. H. Young, *Some characteristic features of twentieth century pure mathematical research.*

L. E. Dickson, *Outline of the theory to date of the arithmetics of algebras.*

S. Pincherle, *Opérations fonctionnelles.*

M. Le Roux, *Sur l'intégration des équations aux dérivées partielles par des intégrales définies.*

The following is a list, as nearly accurate as it has been possible to secure, of the papers presented at the sectional sessions.

Section I. W. Kapteyn, *Expansion of functions in terms of Bernoullian functions*; G. A. Bliss, *The transformation of Clebsch in the calculus of variations*; J. Ouspensky and B. Venkov, *On some new class number relations*; M. Plancherel, *Sur les séries de fonctions orthogonales*; P. A. MacMahon, *Some recent developments in determinant theory*; L. E. Dickson, *Further development of the theory of arithmetics of algebras*; G. Prasad, *On the numerical solution of integral equations*; O. E. Glenn, *Differential combinants and associated parameters*; J. A. Shohat (Jacques Chokhate), *On the asymptotic properties of a certain class of Tchebycheff polynomials*; B. A. Bernstein, *Modular representations of finite algebras*; N. Gunter, *Sur la résolution des systèmes des équations* Grad $X = A$ Rot $X = A$; F. H. Murray, *The asymptotic distribution of the characteristic numbers for the self-adjoint linear partial differential equations of the second order*; J. F. Ritt, *Elementary functions and their inverses*; Th. Varopoulos, *Sur les valeurs exceptionnelles des fonctions multiformes*; L. Pomey, *Les équations intégro-différentielles à plusieurs variables*; O. Öre, *A new method in the theory of algebraic numbers*; H. S. Vandiver, *On the first case of Fermat's last theorem*; R. Birkeland, *On the solution of quintic equations*; W. L. G. Williams, *Formal modular invariants of forms in several variables*; J. Wolff, *On the sufficient conditions for analyticity*; E. T. Bell, *General class number relations whose degenerates involve indefinite forms*; O. E. Glenn, *The method of invariant elements in a modular field*; N. Kryloff, *Sur quelques recherches dans le domaine de la théorie de l'interpolation et des quadratures dites mécaniques*; N. Kryloff and T. Tamarkine, *Sur une formule d'interpolation*; J. Touchard, *Sur la théorie des différences*; J. Touchard, *Sur certaines équations fonctionelles*; J. I. Del Corrail, *Meditations on trigonometry*; A. Lévy, *Sur le calcul des idéaux d'un corps du second degré*; M. Petrovitch, *Correspondance entre la fonction et la fraction décimale*; L. E. Dickson, *A new theory of linear transformations and pairs of bilinear forms*; L. Pomey, *La théorie des nombres*; D. R. Curtiss, *Rational processes for separating the real branches of a plane curve at a multiple point*; T. R. Rosebrugh, *A theorem on the moduli of substitutions involving products of variables themselves given by linear substitutions*; N. Gunter, *Sur un problème fondamental de la hydrodynamique*; E. Narishkina, *On the analogs of Bernoullian numbers in quadratic fields*; J. Drach, *Sur "l'intégration logique" des équations différentielles de la géométrie et de la mécanique*; L. Crelier, *Sur les équations intégrales simples*; E. Hille, *On the zeros of functions defined by linear differential equations*; R. Fueter, *Some applications of the theory of functions to the theory of numbers*; G. C. Evans, *The Dirichlet problem for the general finitely connected open region*; A. Razmadze, *Sur quelques formules de la moyenne*; G. A. Miller, *Commutative conjugate cycles in subgroups of*

the holomorph of an Abelian group; W. Sierpinski, *Les ensembles bien définis non mesurables*; V. Stekloff, *Sur les problèmes de représentation approchée des fonctions à l'aide des polynomes, du calcul approché des intégrales définies, du développement des fonctions en séries de polynomes et de l'interpolation, considérés au point de vue des idées de Tchebycheff*; J. C. Fields, *A basis for the theory of ideals*; B. Delaunay, *Sur le nombre de représentations d'un nombre par une forme binaire cubique d'un discriminant négatif*; R. L. Wilder, *On a certain type of connected set which cuts the plane*; J. I. Hutchinson, *On the roots of the Riemann zeta function*; W. B. Ford, *On determining the asymptotic developments of a given function*; L. Tonelli, *Sul calcolo delle variazioni*; A. Razmadze, *Sur les extrémales discontinues dans le calcul des variations*; L. G. Du Pasquier, *L'évolution du concept du nombre hypercomplexe entier*; T. Andrade, *Problème proposé sur les équations fonctionnelles*; O. C. Hazlett, *On the arithmetic of a general associative algebra*; M. Kossler, *A generalization of the theorems of Fabry and Szasz*; M. Kravtschok, *Note sur l'interpolation généralisée.*

Section II. C. Servais, *Sur les lignes asymptotiques*; E. Cartan, *Sur la déformation projective des surfaces*; A. B. Coble, *The behavior of the rational plane sextic and its related Cayley symmetroid under regular Cremona transformation*; T. Bonnesen, *Recherches géométriques sur le problème isopérimétrique*; J. H. Weaver, *On a system of triangles related to a poristic system*; J. A. Barrau, *Conditions of intersection of flat spaces situated in a quadratic variety*; F. Morley, *The condition that the curves of a net have a common point*; M. Fréchet, *Expression la plus générale de la "distance" sur une droite*; G. Fubini, *Principles of projective differential geometry*; A. R. Da Costa, *L'enseignement des mathématiques doit être orienté pour l'étude de la relativité*; A. Naess, *On a method of generalizing the vector products to* S_m; A. Naess, *Three theorems of analysis derived by the vector method as corollaries from a single proposition*; F. D. Murnaghan, *The generalized Kronecker symbol*; N. Delaunay, *Sur les bases nouvelles de la théorie des systèmes articulés*; G. Tzitzeica, *Un nouveau problème sur les suites de Laplace*; M. Fréchet, *Sur une représentation paramétrique intrinsèque de la courbe continue la plus générale*; C. H. Sisam, *On surfaces whose asymptotic curves are cubics*; E. A. L. Merlin, *Sur les réseaux asymptotiques*; B. Delaunay, *Sur la sphère vide*; M. Janet, *Sur les systèmes linéaires d'hypersurfaces*; G. Koenigs, *Sur les mouvements à deux paramètres doublement décomposables*; C. Servais, *Sur la géométrie du tétraèdre*; N. B. MacLean, *On certain surfaces related covariantly to a given ruled surface*; L. A. Godeaux, *Sur les involutions régulières d'ordre deux appartenant à une surface irrégulière*; C. T. Sullivan, *The determination of all surfaces characterized by a reducible directrix quadric*; A. Errera, *Quelques remarques sur le problème*

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Section III. S. Zaremba, *Sur un groupe de transformations qui se présente en électrodynamique*; W. F. G. Swann, *A new deduction of the electromagnetic equations*; P. Haag, *Sur un problème général de probabilités et ses diverses applications*; C. V. Raman, *Theory of the structure of liquid surfaces*; E. W. Brown, *The orbit of the eighth satellite of Jupiter*; L. Silberstein, *A finite world radius and some of its cosmological implications*; S. Chapman, *Steps towards a new theory of magnetic storms*; R. H. Fowler, *The equilibrium properties of gases at high (stellar) temperatures*; C. Snow, *Alternating current distribution in cylindrical conductors*; A. W. Conway, *On the quantisation of certain orbits*; P. Haag, *Sur l'application des méthodes du calcul tensoriel à la théorie des moindres carrés*; G. Greenhill, *The spinning top in two moves*; G. Giorgi, *The functional dependence of physical variables*; A. C. Dixon, *A rectangular plate, clamped at edges*; J. Gianfrancheschi, *The perturbations in electrons' orbits*; H. M. Dadourian, *On the fundamental principle of dynamics*; S. J. Jacobsohn, *Note on the force equation of electrodynamics*; W. J. Humphreys, *The effect of surface drag on surface winds*; G. F. McEwen, *Calculation of the velocity of vertical ocean currents in the San Diego region from the accompanying temperature reduction below "normal" values*; A. H. S. Gillson, *The dynamical theory of tides in an ocean of varying depth*; N. H. Heck, *Velocity of sound in sea water*; D. Buchanan, *Asymptotic solutions in the problem of three bodies*; J. Patterson, *The theory of the anemometer*; H. C. Plummer, *Note on the reduction of parallax plates*; P. Haag, *Sur le problème des séquences*; L. V. King, *On the direct numerical calculation of elliptic functions and integrals*; D. Wrinch, *A subject in hydrodynamics*; S. Chapman, *The electrostatic*

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ARNOLD DRESDEN

A THEOREM ON SIMPLE ALGEBRAS*

BY J. H. M. WEDDERBURN

In a previous paper† I showed that every simple algebra A can be expressed as the direct product of a division algebra D and a simple matric algebra $M = (e_{pq})$; the object of this note is to show that this expression is unique, that is, *if* $A = D_1 \times M_1 = D_2 \times M_2$, *where* D_1 *and* D_2 *are division algebras and* M_1 *and* M_2 *are simple matric algebras, then* D_1 *and* M_1 *are simply isomorphic‡ with* D_2 *and* M_2 *respectively.*

Let δ_1 and δ_2 be the orders of D_1 and D_2, and let e_1 and e_2 be primitive idempotent elements of M_1 and M_2 respectively. If e_1 and e_2 are supplementary or equal, then§

$$D_1 \cong e_1 A e_1 \simeq e_2 A e_2 \cong D_2;$$

M_1 and M_2 are then of the same order and are therefore simply isomorphic. We shall therefore suppose that $e_1 \neq e_2$ and, say, $e_1 e_2 \neq 0$.

Assume in the first place that $x = e_1 e_2$ is not nilpotent; there then exists a rational polynomial

$$y = \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_r x^r,$$

which is an idempotent element of the algebra X generated by x. Now $e_1 x = x$; therefore, since every element of X has the form $xf(x)$, $f(x)$ a polynomial in x, it follows that $e_1 y = y$ and

$$(ye_1)^2 = ye_1 ye_1 = y^2 e_1 = ye_1 = e_1 ye_1,$$

so that $ye_1 \leqq e_1 A e_1$; also $ye_1 \neq 0$ since $ye_1 y = y^3 = y \neq 0$; hence ye_1, being idempotent, equals e_1. In the same way

* Presented to the Society, May 3, 1924.

† PROCEEDINGS OF THE LONDON SOCIETY, (2), vol. 6 (1907), p. 99.

‡ *Simple isomorphism* will be denoted by $\cong$.

§ See L. E. Dickson, *Algebras and Their Arithmetics*, Chicago, 1923, pp. 74, 77.

it follows that $e_2y = e_2$ and $ye_2 = y$. Also, if we set $y_{10} = y - e_1$, $y_{02} = y - e_2$, then

$$e_1y_{10} = y_{10},\ y_{10}e_1 = 0,\ e_2y_{02} = 0,\ y_{02}e_2 = y_{02},$$

and, since $e_1 \neq e_2$, one of y_{10}, y_{02} is not zero, say y_{10}.

We shall now show that y is primitive in A. Since $e_1y = y$, every element of yAy has the form*

$$w = d_{11}e_{11} + d_{12}e_{12} + \cdots, \quad (d_{ij} < D_1;\ e_{11} = e_1,\ e_{ij} < M_1).$$

If w is idempotent, this gives $d_{11} = a$, the modulus of A, so that, if w is primitive, $y - w$ lacks the term in e_{11} and, being therefore nilpotent, must equal 0 since $(y-w)^2 = y - w$. Hence y is primitive and we may set† $A = D \times M$ where M is a simple matric algebra which contains y and D is a division algebra simply isomorphic with yAy.

Now, since $ye_1 = e_1$,

$$De_1 = Dye_1 = yAye_1 \geqq ye_1Ae_1ye_1 = e_1Ae_1 = D_1e_1$$

also

$$D_1e_1 = e_1Ae_1 \geq e_1yAye_1 = yAye_1 = Dye_1 = De_1;$$

hence $De_1 = D_1e_1$ and therefore $D \cong D_1$. Similarly $D \cong D_2$; hence $D_1 \cong D_2$ and, as before, also $M_1 \cong M_2$.

Suppose in the second place that e_1e_2 is nilpotent; then $(e_1e_2e_1)^r = (e_1e_2)^re_1$ is also nilpotent and, as e_1Ae_1 is a division algebra, it follows that $e_1e_2e_1 = 0$; hence $(e_2e_1)^2 = 0$ and so, by a repetition of the same argument, $e_2e_1e_2 = 0$. If now we set $y = e_1 - e_1e_2 - e_2e_1$, then

$$e_1y = e_1 - e_1e_2 \neq e_1,\ ye_1 = e_1 - e_2e_1,\ e_1ye_1 = e_1,$$
$$e_2y = 0 = ye_2,\ ye_1y = y,\ y^2 = y.$$

As before we must show that y is primitive. If $y = y_1 + y_2$ where y_1 and y_2 are supplementary idempotent elements, then

$$0 = e_2y = e_2y_1 + e_2y_2,$$

so that

$$0 = (e_2y_1 + e_2y_2)y_1 = e_2y_1,$$

and similarly

$$y_1e_2 = e_2y_2 = y_2e_2 = 0.$$

* See Dickson, loc. cit.

† See Dickson, loc. cit.

From $y_1+y_2=y=e_1-e_1e_2-e_2e_1$ we have therefore

$$y_1=yy_1=e_1y_1-e_2e_1y_1,$$

so that $y_1e_1y_1=y_1$, and similarly

$$y_2e_1y_2=y_2,\ y_2e_1y_1=y_2y_1=0=y_1e_1y_2.$$

Now, if $z_1=e_1y_1e_1$, $z_2=e_1y_2e_1$, then

$$e_1=e_1ye_1=e_1y_1e_1+e_1y_2e_1=z_1+z_2,$$
$$z_1^2=e_1y_1e_1\cdot e_1y_1e_1=e_1\cdot y_1e_1y_1\cdot e_1=e_1y_1e_1=z_1,$$
$$z_1z_2=e_1y_1e_1\cdot e_1y_2e_1=e_1y_1e_1y_2e_1=0,$$

and similarly $z_2^2=z_2$, $z_2z_1=0$. Also $z_1\neq 0$ unless $y_1=0$ since

$$y_1z_1y_1=y_1e_1y_1e_1y_1=y_1e_1y_1=y_1,$$

and similarly $z_2\neq 0$ unless $y_2=0$. But z_1 and z_2, if not zero, are supplementary idempotent elements in e_1Ae_1 whereas e_1 is primitive; hence one of them is zero, that is, y is primitive.

We may now, as before, set $A=D\times M$ where $D\cong yAy$ is a division algebra and M is a simple matric algebra containing y. Remembering that $e_1ye_1=e_1$, we then have

$$e_1Dye_1=e_1yAye_1\geqq e_1ye_1Ae_1ye_1=e_1Ae_1=D_1e_1,$$

also

$$D_1e_1=e_1Ae_1\geqq e_1yAye_1=e_1Dye_1;$$

hence $D_1e_1=e_1Dye_1$. If d and d' are any elements of D, we therefore have

$$e_1dye_1=d_1e_1,\ e_1d'ye_1=d_1'e_1,\qquad (d_1,\ d_1'<D_1)$$

and hence

$$e_1(d+d')ye_1=(d_1+d_1')e_1,$$
$$d_1d_1'e_1=e_1dye_1d'ye_1=e_1dye_1yd'e_1=e_1dyd'e_1=e_1dd'ye_1,$$

and, since $e_1=e_1ye_1$, it follows that $D\cong D_1$ and therefore $M\cong M_1$. Finally, since $e_2y=0=ye_2$, e_2 and y are supplementary and hence $D_2\cong e_2Ae_2\cong yAy\cong D\cong D_1$, from which, as before, $M_1\cong M_2$. The proof of the theorem is therefore complete.

PRINCETON UNIVERSITY

NOTE ON A CLASS OF HARMONIC FUNCTIONS

BY G. C. EVANS

1. *Introduction.* Let S be a multiply-connected open region, bounded by an exterior circle s_0 and n interior circles $s_1, \cdots, s_n$, not touching, and let $u(P)$ be harmonic, not necessarily bounded, in S. The author has shown that a necessary and sufficient condition that $u(P)$ be given as the sum of logarithmic terms and Poisson-Stieltjes integrals around the several boundaries is that the integrals $\int_0^{2\pi} |u| d\theta_i$, extended around circles concentric with each boundary circle, remain bounded in the neighborhood of each boundary circle. He has shown also that a necessary and sufficient condition that $u(P)$ be given by the corresponding Stieltjes integral generalization of the formula in terms of the normal derivative of the Green's function is that $\int |u| dh$ remain bounded, where the integration is extended over the whole of a variable curve $g = \text{const.}$, with g the Green's function and h its conjugate, referred to some fixed pole Q. Finally the author has shown that a necessary and sufficient condition for the latter formula is that $u(P)$ be the difference of two functions harmonic and not-negative in S.* It is desired to show that this condition is also necessary and sufficient for the former mode of representation, so that the two forms of representation apply to the same class of functions. That the condition is necessary follows at once from the mode of representation. It is only the sufficiency that requires proof.

2. *Lemma* I. Let S_0 be a simply-connected open region, bounded internally or externally by a circle s_0, of radius

* *The Dirichlet problem for the general open finitely-connected region,* PROCEEDINGS OF THE INTERNATIONAL CONGRESS AT TORONTO (1924).

R_0, and let $u(P)$ be the difference of two not-negative harmonic functions $u'(P)$, $u''(P)$ in an annular neighborhood in S_0 of the boundary s_0. Then $\int_0^{2\pi}|u|d\theta$ is bounded for the circles of that neighborhood concentric with s_0.

For convenience let S_0 be an interior region. Let s', of radius r', be a fixed circle in the annular neighborhood and let s be a circle of radius r, $r' \leqq r < R_0$, all these circles having the same center 0. The lemma follows immediately from Bôcher's device, evaluating the surface integral of $(1/r)\partial u/\partial r$. In fact, by integrating this expression in u' over the annular region between s' and s, and equating its two evaluations as an iterated integral, we have the equation

$$\left(\int_0^{2\pi} u' d\partial\right)_r = \left(\int_0^{2\pi} u' d\theta\right)_{r'} + \alpha_1 \log r - \alpha_1 \log r',$$

where α_1 is a constant. Hence

$$\int u' d\theta = \int |u'| d\theta$$

is bounded. Similarly $\int |u''| d\theta$ is bounded, and therefore $\int |u| d\theta$ is bounded.

3. *Lemma* II. In the open region S of § 1, let $u(P)$ be the difference of two functions $u'(P)$, $u''(P)$, harmonic and not-negative in S. Then $u(P)$ is the sum of $n+1$ functions $u_i(P)$, each harmonic except for possible logarithmic singularities at infinity, in the simply-connected open region S_i bounded by s_i, and each satisfying in its region S_i the conditions of Lemma 1.

The first part of this lemma is proved by Osgood.* The last part follows immediately by writing, in the neighborhood of s_i, $u_i(P)$ as the difference $u(P) - \overline{u}(P)$ where

$$\overline{u}(P) = \sum_j' u_j(P),$$

the summation not including $j = i$.

* Osgood, *Lehrbuch der Funktionentheorie*, Leipzig and Berlin, 1912, pp. 642–644.

4. *Proof of the Theorem.* The statement § 1 can now be proved. In fact, if the hypothesis of Lemma 2 holds, $\int |u_i| d\theta_i$ is bounded for circles concentric with s_i in the annular neighborhood in S of s_i, by Lemma 1. The same is true for $\int |u_j| d\theta_i, j \neq i$, since $u_j(P)$ is harmonic on s_i itself if j is unequal to i. But

$$\int |u| d\theta_i \leqq \sum_{j=0}^{n} \int |u_j| d\theta_i.$$

Hence the condition of the first cited theorem of § 1 is satisfied. This is what was to be proved.

4. *Conclusion.* If we make a conformal transformation of S into a general open region T of the same connectivity, without isolated point boundaries, the Toronto memoir already cited shows that the condition that $u(P)$ be the difference of two not-negative harmonic functions is necessary and sufficient that $u(P)$ be given by a generalized Stieltjes integral in terms of the conjugate to the Green's function for T on the ordered boundary points of T. But if we make conformal transformations of each S_i into the corresponding simply-connected region T_i, we obtain a representation of u as the sum of terms u_i where each u_i is the sum of a possible logarithmic term and a generalized Stieltjes integral on the frontier points of T_i, expressed in terms of the conjugate to the Green's function for T_i. And from what we have proved in § 3, it follows that this representation applies to precisely the same class of harmonic functions as the other. The Dirichlet problem is a special case of the problem here discussed.

The connection between the two forms of representation of $u(P)$ is given by integral equations which are worth study on their own account.

THE RICE INSTITUTE

THREE THEOREMS ON NORMAL ORTHOGONAL SETS

BY M. H. STONE

The three theorems which we desire to record here may be of some interest, in spite of their simplicity. So far as the author knows, they have not appeared in the literature. On the other hand, the corollaries appended are to be found in the journals cited in the footnotes, but were there obtained in different fashion. Throughout our work we shall assume on the part of the reader some familiarity with the theorem of Riesz-Fischer and with convergence in the mean.* All functions considered are understood to be real functions defined on the interval (a, b). They are to be summable with summable square in the Lebesgue sense on this interval. The first result is embodied in the following theorem.

THEOREM I. *A necessary and sufficient condition that every pair of functions f_1, f_2 having identical Fourier coefficients with respect to a normal orthogonal set $\{u_k\}$ should have identical coefficients with respect to a second set $\{v_k\}$ is that the formal series for each function of the set $\{v_k\}$ in terms of $\{u_k\}$ should converge in the mean to that function.*

The condition is necessary. We write

$$\int_a^b v_i u_k = c_{ik},$$

so that the formal series for v_i is $\sum_{k=1}^{\infty} c_{ik} u_k$. Suppose that for some fixed i the series does not converge in the mean

* Plancherel, RENDICONTI DI PALERMO, vol. 30 (1910), pp. 289–335, especially pp. 290–297. See also Riesz, GÖTTINGER NACHRICHTEN, 1907, p. 117.

to v_i; that is, that

$$\sum_{k=1}^{\infty} c_{ik}^2 < \int_a^b v_i^2.$$

By the Riesz-Fischer theorem, the series converges in the mean to a second function v_i^* for which we have

$$\int_a^b v_i^* u_k = c_{ik}$$

for all k. Now

$$\lim_{K\to\infty} \int^b \left(v_i^* - \sum_{k=1}^{K} c_{ik} u_k\right) v_i = 0$$

as we see by the use of Schwarz's inequality. In other words

$$\int_a^b v_i^* v_i = \sum_{k=1}^{\infty} c_{ik}^2 + \int_a^b v_i v_i.$$

The functions v_i and v_i^* have identical coefficients with respect to $\{u_k\}$ but different coefficients with respect to the set $\{v_k\}$. This contradicts the hypothesis, and therefore shows the necessity of the condition.

The condition is sufficient. We write $\int_a^b f_1 u_k = \int_a^b f_2 u_k = a_k$. We use the Schwarz inequality to prove that

$$\lim_{K\to\infty} \int_a^b f_1 \left(v_i - \sum_{k=1}^{K} c_{ik} u_k\right) = 0,$$

$$\lim_{K\to\infty} \int_a^b f_2 \left(v_i - \sum_{k=1}^{K} c_{ik} u_k\right) = 0.$$

It follows that $\int_a^b f_1 v_i = \int_a^b f_2 v_i = \sum_{k=1}^{\infty} c_{ik} a_k$. This completes the proof of Theorem I.

We shall next prove the following theorem.

THEOREM II. *A necessary and sufficient condition that the formal Fourier series* $\sum_{k=1}^{\infty} a_k u_k$, $\sum_{k=1}^{\infty} b_k v_k$ *associated with an arbitrary function* f *should converge in the mean to the same function* f^* *is that the formal series for each function of the set* $\{v_k\}$ *in terms of the set* $\{u_k\}$ *should converge in the mean to that function and vice versa.*

The condition is necessary. This fact appears as obvious as soon as we set $f = u_k$, $f = v_k$, $(k = 1, 2, \cdots, \infty)$.

The condition is sufficient. The series $\sum_{k=1}^{\infty} a_k u_k$ converges in the mean to a function f_1^*, the series $\sum_{k=1}^{\infty} b_k v_k$ to a function f_2^*, by the theorem of Riesz-Fischer. We now appeal to Theorem I to show that

$$\int_a^b f u_k = \int_a^b f_1^* u_k = \int_a^b f_2^* u_k, \quad (k = 1, 2, \cdots, \infty).$$

Since f, f_2^* have identical coefficients with respect to $\{v_k\}$, they also have identical coefficients with respect to $\{u_k\}$ by Theorem I. The equalities above then result. In the same manner we show that

$$\int_a^b f v_k = \int_a^b f_1^* v_k = \int_a^b f_2^* v_k, \quad (k = 1, 2, \cdots, \infty).$$

Next, from Bessel's inequality, we obtain

$$\sum_{k=1}^{\infty} a_k^2 = \int_a^b f_1^{*2} \geq \sum_{k=1}^{\infty} b_k^2, \quad \sum_{k=1}^{\infty} b_k^2 = \int_a^b f_2^{*2} \geq \sum_{k=1}^{\infty} a_k^2.$$

The inequalities are therefore impossible. We see that either of the two series $\sum_{k=1}^{\infty} a_k u_k$, $\sum_{k=1}^{\infty} b_k v_k$ converges in the mean to each of the two functions f_1^*, f_2^*. The two functions are therefore the same, in the sense that they can differ at most on a set of zero measure. This completes the proof of Theorem II.

The two corollaries which we shall now discuss are to be found in the literature.

COROLLARY I.* *A necessary and sufficient condition that a normal orthogonal set $\{u_k\}$ be closed is that the formal series for each function of an arbitrary closed normal orthogonal set $\{v_k\}$ in terms of $\{u_k\}$ converge in the mean to that function.*

In order that $\{u_k\}$ be a closed set, it is necessary and sufficient that every function represented by the convergence

* Lauricella, RENDICONTI DEI LINCEI, (5), vol. 21 (1912), pp. 675–85.

in the mean of its formal series in terms of the closed set $\{v_k\}$ be so represented in terms of $\{u_k\}$. Theorem II then gives us the last link in the proof.

COROLLARY II.* *Let $\{\Phi_k\}$ be a set of functions such that the system of equations $\int_a^b \theta\Phi_k = 0$, $k = 1, 2, \cdots, \infty$, implies that the function θ is identically zero except possibly on a set of zero measure. Then a necessary and sufficient condition that a normal orthogonal set $\{u_k\}$ be closed is that the formal series for each function of the set $\{\Phi_k\}$ in terms of $\{u_k\}$ converge in the mean to that function.*

By Schmidt's process of normalization and orthogonalization we form from the set $\{\Phi_k\}$ a closed normal orthogonal set $\{v_k\}$. Each function of $\{v_k\}$ is a linear combination with constant coefficients of a finite set of functions of $\{\Phi_k\}$, and conversely. We recall that series which can be summed in the sense that they converge in the mean can be added term by term. By using Corollary I in conjunction with this fact we complete our demonstration.

Of particular interest under the second corollary is the special case obtained by setting $\Phi_k = x^{k-1}$, $k = 1, 2, \cdots, \infty$. For further remarks on these two corollaries we refer to the papers already mentioned.

A theorem very similar to Theorem II follows.

THEOREM III. *A necessary and sufficient condition that the difference of the two formal series $\sum_{k=1}^{\infty} a_k u_k$, $\sum_{k=1}^{\infty} b_k v_k$ associated with an arbitrary function f should converge in the mean to zero, is that the formal series for each function of $\{v_k\}$ in terms of $\{u_k\}$ converge in the mean to that function, and vice versa.*

The condition is necessary. We can see this fact readily by putting $f = u_k$, $f = v_k$, $(k = 1, 2, \cdots, \infty)$.

The condition is sufficient. We know by Theorem II that the two series converge in the mean to the same

* Severini, RENDICONTI DI PALERMO, vol. 36 (1913), pp. 177–202, especially pp. 194–95.

function f^*. Hence their difference converges in the mean to zero.

This last theorem may be looked upon as giving a necessary and sufficient condition that two normal orthogonal sets $\{u_k\}$, $\{v_k\}$ be equivalent in the sense that the formal series for an arbitrary function in terms of them always converge in the mean to the same function.

NEW YORK CITY

ON THE COMPLETE INDEPENDENCE OF THE FUNCTIONAL EQUATIONS OF INVOLUTION*

BY C. C. MAC DUFFEE

1. *Introduction.* The three fundamental theorems or "laws" of involution are commonly written

$$\text{I } a^m \cdot a^n = a^{m+n}, \quad \text{II } (a^m)^n = a^{mn}, \quad \text{III } a^n \cdot b^n = (ab)^n.$$

In §2 of this paper these equations are abstractly formulated as functional equations. In §3 it is proved that any function satisfying Equations I and II also satisfies III, provided certain underlying conditions A and B are fulfilled.

In § 4 a number system $\mathfrak{N}$ is introduced whose elements are the numbers $[\xi, r]$ where ξ and r are real numbers. Two operations, addition and multiplication, are introduced, and it is shown that these operations obey all the ordinary laws of algebra except the associative law of addition. This number system $\mathfrak{N}$ is then used in the discussion of the complete independence† of Equations I, II and III.

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* Presented to the Society, April 28, 1923.

† Consider n conditions. There are conceptually 2^n distinct cases to be considered according as Condition 1 holds or does not hold, Condition 2 holds or does not hold, $\cdots$, Condition n holds or does not hold. If none of these 2^n cases is empty, the n conditions are said to be completely independent. Cf. E. H. Moore, THE NEW HAVEN MATHEMATICAL COLLOQUIUM, (Yale University Press, 1910), pp. 81, 82.

Subject to the conditions of § 2 the functional equations are not completely independent, for I and II imply III. However, the remaining 7 cases are non-empty, for in § 5 of this paper 7 functions are defined which obey the conditions of § 2, each fulfilling one of these 7 cases.

The writer wishes to acknowledge his indebtedness to Professor E. H. Moore of the University of Chicago and to Professor E. V. Huntington of Harvard University for suggestions and criticisms.

2. *A General Formulation of the Functional Equations.* Consider a system Σ composed of the following terms:

(1) Two classes C, C' whose elements will be denoted respectively by c, c'.

(2) Three single-valued functions* or operations:

$\times$ on CC to C, $\quad \times'$ on $C'C'$ to C', $\quad +'$ on $C'C'$ to C'.

(3) A single-valued function φ on CC' to C.

We assume that the following underlying conditions are satisfied:

A. The right-hand distributive law holds for C', namely

$$(c_1' +' c_2') \times' c_3' = (c_1' \times' c_3') +' (c_2' \times' c_3')$$

for every c_1', c_2', c_3'.

B. There is an element c_0 of C such that for every c there is a c' such that $c = \varphi(c_0, c')$.

We now consider the following postulates in connection with system Σ:

I. $\quad \varphi(c, c_1') \times \varphi(c, c_2') = \varphi(c, c_1' +' c_2')$,

II. $\quad \varphi(\varphi(c, c_1'), c_2') = \varphi(c, c_1' \times' c_2')$,

III. $\quad \varphi(c_1, c') \times \varphi(c_2, c') = \varphi(c_1 \times c_2, c')$.

It is understood that these relations hold for all values of the arguments.

As an example of a system Σ in which Conditions *A*

*Thus for example, if c_1, c_2 are two elements of C, $\times(c_1, c_2)$ or $c_1 \times c_2$ denotes a definite element c_3 of C.

and B and all three postulates are satisfied, we may identify the class C with the class of all real numbers $\geqq 1$, C' with the class of all numbers $\geqq 0$, $\times$ and $\times'$ with the ordinary operation of multiplication, $+'$ with ordinary addition, and $\varphi(\ , c')$ with the power function $c^{c'}$.

3. THEOREM. *If system Σ satisfies Conditions A and B, and φ satisfies Postulates* I *and* II, *then φ also satisfies Postulate* III.

Suppose that c_0 is the element c_0 of Condition B, and that c_1 and c_2 are any two elements of C. Then by Condition B elements c_1' and c_2' exist so that

$$(1) \qquad c_1 = \varphi(c_0, c_1'), \qquad c_2 = \varphi(c_0, c_2').$$

Therefore we have

$$\begin{aligned}
\varphi(c_1 \times c_2, c') &= \varphi(\varphi(c_0, c_1') \times \varphi(c_0, c_2'), c'), && (1),\\
&= \varphi(\varphi(c_0, c_1' +' c_2'), c'), && (I),\\
&= \varphi(c_0, (c_1' +' c_2') \times' c'), && (II),\\
&= \varphi(c_0, c_1' \times' c' +' c_2' \times' c'), && (A),\\
&= \varphi(c_0, c_1' \times' c') \times \varphi(c_0, c_2' \times' c'), && (I),\\
&= \varphi(\varphi(c_0, c_1'), c') \times \varphi(\varphi(c_0, c_2'), c'), && (II),\\
&= \varphi(c_1, c') \times \varphi(c_2, c'), && (1),
\end{aligned}$$

which is precisely Postulate III.

4. *An Algebra Whose Elements are Pairs of Real Numbers.* Consider the class of all "numbers" or number pairs of the type $[\xi, r]$ where ξ and r are real numbers. The equality

$$[\xi_1, r_1] = [\xi_2, r_2]$$

implies that $r_1 = r_2$, and unless $r_1 = r_2 = 0$ it also implies that $\xi_1 = \xi_2$. The number $[\xi, 0]$ is supposed to be independent of ξ and will be called *zero*.

When $r_1 r_2 \neq 0$, addition of two such number pairs is defined by the identity

$$[\xi_1, r_1] + [\xi_2, r_2] = [\xi_3, r_3]$$

where r_3 and ξ_3 are defined by the relations

$$r_3 \sin \xi_3 = r_1 \sin \xi_1 + r_2 \sin \xi_2,$$
$$r_3 \cos \xi_3 = r_1 \cos \xi_1 + r_2 \cos \xi_2,$$
$$\frac{\xi_1+\xi_2}{2} - \frac{\pi}{2} < \xi_3 \leqq \frac{\xi_1+\xi_2}{2} + \frac{\pi}{2}.$$

When either r_1 or r_2 or both are zero, addition is defined by the relations

$$[\xi_1, 0]+[\xi_2, r_2] = [\xi_2, r_2], \qquad [\xi_1, r_1]+[\xi_2, 0] = [\xi_1, r_1].$$

The sum always exists uniquely.

The operation of multiplication is defined by the identity

$$[\xi_1, r_1]\times[\xi_2, r_2] = [\xi_1+\xi_2, r_1 r_2].$$

The product always exists uniquely.

The algebra of such number pairs subject to the operations of addition and multiplication as defined above we shall call the algebra $\mathfrak{N}$. It is evident that both addition and multiplication in $\mathfrak{N}$ are commutative, for the identities which define these operations are symmetric in ξ_1, r_1 and ξ_2, r_2. It may be shown directly that multiplication is associative and distributive with respect to addition. It is worthy of note however that addition is not always associative. This fact is sufficient to show that $\mathfrak{N}$ is not isomorphic with any linear algebra.

5. *Concerning the Complete Independence of the Postulates.* In connection with three conditions 1, 2, 3 there are conceptually eight distinct cases to be considered, each case being defined by the holding or non-holding of each condition. If none of these cases is empty, the three conditions are completely independent. We wish to consider the complete independence of Postulates I, II, III of § 2 in a system Σ with Conditions A and B holding. To settle this question completely we must consider eight cases, and the existence of eight functions φ_{ijk} where i is 1 or 0 according as the function φ_{ijk} satisfies or does not satisfy I, j is 1 or 0 according as it satisfies or does not satisfy II, and k is 1 or 0 according as it satisfies or does not satis-

fy III. In view of the theorem of § 3 no function φ_{110} can exist, for every function φ in Σ satisfying A, B, I and II must necessarily satisfy III. We show that no further dependence theorems are possible by exhibiting examples of functions in each of the remaining seven cases.

In each of the following examples we shall use $\mathfrak{R}$ as the class C', and $\mathfrak{R}$ or a subclass of $\mathfrak{R}$ which is closed under the $\times$-operation as the class C. We shall denote by C_1 the subclass of $\mathfrak{R}$ composed of all numbers $[\xi, r]$ for which $r > 0$; by C_2 the subclass of $\mathfrak{R}$ for which $\xi = 0$ and $r > 0$; by C_3 the subclass for which $r = 1$; and by C_4 the subclass composed of zero and all the other numbers of $\mathfrak{R}$ for which $\xi > 0$. Evidently all four of these subclasses are closed under the $\times$-operation.

By considering in connection with C and C' a single-valued function φ_{ijk}, we have a system Σ. Moreover, since the distributive law holds for $\mathfrak{R}$, Condition A is satisfied. It will be necessary to verify Condition B for each case, to show that $[\zeta, p] = \varphi_{ijk}([\xi, r], [\eta, s])$ is independent of η if $s = 0$, and then investigate whether each of the three postulates I, II, III is or is not satisfied by φ_{ijk} for all values of its arguments.

Case 111. In addition to the example given in § 2, we may mention the function

$$\varphi_{111}([\xi, r], [\eta, s]) \equiv [\zeta, p]$$

on C_1C' to C_1 where

$$\zeta = s\xi \cos \eta + s \sin \eta \log r, \quad \log p = s \cos \eta \log r - s\xi \sin \eta.$$

If we choose $[0, e]$ to be the element c_0 of Condition B, we find that we may choose as c' any element $[\eta, s]$ defined by the relations $s \sin \eta = \zeta$, $s \cos \eta = \log p$. Hence c' always exists in C' for every element $[\zeta, p]$ in C_1.

When $s = 0$, $[\zeta, p] = [0, 1]$ is independent of η. It may be shown by a direct calculation that φ_{111} satisfies Postulates I, II and III. In fact, this function corresponds to the power function of complex variable theory. The

latter however has a denumerable infinity of values, whereas the function φ_{111} on C_1C' to C_1 is single-valued.

Case 011. The function φ_{011} on C_2C' to C_2 is defined by the identity

$$\varphi_{011}([0, r], [\eta, s]) \equiv [\zeta, p] \equiv [0, r^s].$$

If we take $c_0 = [0, e]$, then $c' = [\eta, \log p]$ where η is arbitrary.

Case 101. The function φ_{101} on C_1C' to C_1 is defined by the identity

$$\varphi_{101}([\xi, r], [\eta, s]) \equiv [\zeta, p] \equiv [s\xi \cos \eta + s \sin \eta \log r, r^{s \cos \eta}].$$

If $c_0 = [0, e]$, then c' is any element $[\eta, s]$ such that

$$s \sin \eta = \zeta, \qquad s \cos \eta = \log p.$$

Case 110 is empty in view of the theorem of § 3.

Case 001. The function φ_{001} on C_3C' to C_3 is defined as

$$\varphi_{001}([\xi, 1], [\eta, s]) \equiv [\zeta, 1] \equiv [\xi s\eta, 1].$$

If $c_0 = [1, 1]$, then c' is any element $[\eta, s]$ such that $\eta s = \zeta$.

Case 010. The function φ_{010} on C_4C' to C_4 is defined as

$$\varphi_{010}([\xi, r], [\eta, s]) \equiv [\zeta, p] \equiv [\xi e^{\eta}, rs].$$

If $c_0 = [e, 1]$, then $c' = [\log \zeta - 1, p]$.

Case 100. The function φ_{100} on C_1C' to C_1 is defined as

$$\varphi_{100}([\xi, r], [\eta, s]) \equiv [\zeta, p] \equiv [s \sin \eta, r\xi^{s \cos \eta}].$$

If $c_0 = [1, e]$, then c' is any element $[\eta, s]$ defined by

$$s \sin \eta = \zeta, \qquad s \cos \eta = \log p.$$

Case 000. For the function φ_{000} on $\mathfrak{R}C$ to $\mathfrak{R}$, we may take

$$\varphi_{000}([\xi, r], [\eta, s]) \equiv [\zeta, p] \equiv [\eta, s].$$

Let c_0 be any convenient element of $\mathfrak{R}$. Then $c' = [\zeta, p]$.

Thus the problem of the complete independence of these postulates is completely solved, subject to the mild restrictions of Conditions *A* and *B*. It is found that I *and* II *jointly imply* III, *and no other dependence relations exist.*

OHIO STATE UNIVERSITY

THE FREQUENCY LAW OF A FUNCTION OF ONE VARIABLE*

BY E. L. DODD

1. *Introduction.* If the probalility that a variable X will take on a value not greater than x is

$$\Phi(x) = \int_{-\infty}^{x} \varphi(x)\,dx,$$

then $\Phi(x)$ is the cumulative frequency law, "Verteilung", for X; whereas the frequency law is $\Phi'(x)$, which is equal to $\varphi(x)$ at a point of continuity of $\varphi(x)$. Two closely related problems will be treated in this paper:

(1) Given the frequency law $\varphi(x)$ for a variable X, to find the frequency law $\psi(y)$ for a function Y of X;

(2) Given $\varphi(x)$ and $\psi(y)$, to find Y.

Under (1), where $Y = f(X)$ is given as a continuous increasing function, no special difficulty arises.† When, however, $f(X)$ has an infinitely multiple-valued inverse $g(Y)$, the expression naturally assignable to $\psi(y)$ will not be valid without restriction.

Two real functions $\varphi(x)$ and $f(x)$ will be introduced defined for all real values of x. In case a given $\varphi(x)$ or $f(x)$ is undefined outside a finite interval, the value zero may be assigned to it outside this interval.

As the foregoing problems belong essentially to general analysis, the two theorems to be stated will avoid the language of probability.

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* Presented to the Society, December 30, 1924.

† Mayr, *Wahrscheinlichkeitsfunktionen und ihre Anwendungen*, Monatshefte für Mathematik und Physik, vol. 30 (1920), pp. 17–43. Rietz, *Frequency distributions obtained by certain transformations of normally distributed variates*, Annals of Mathematics, (2), vol. 23 (1922), pp. 292–300.

2. *The Determination of the Frequency Law for a Function.*

THEOREM I. *Let*

$$z = \varphi(x) \tag{1}$$

be a single-valued real function of x, integrable over the real continuum. Let

$$y = f(x) \tag{2}$$

be a continuous single-valued real function, defined for all real values of x, with an inverse function,

$$x = g(y), \tag{3}$$

in general multiple-valued. In a formal manner, set

$$\psi(y) = \sum \varphi[g(y)]\ |g'(y)|, \tag{4}$$

where a term is to appear in the sum for each x corresponding to the given y, and where $\psi(y) = 0$ if there is no such x; and set

$$\Psi(\eta) = \int_{f \leqq \eta} \varphi(x)\, dx, \tag{5}$$

this definite integral to be taken over those portions of the x-axis for which $f(x) \leqq \eta$, and to be zero if $f(x) > \eta$ identically. Given a real number η for which $f(x) - \eta$ has at least one zero, suppose that $\varphi[g(\eta)]$ is continuous for each corresponding x. Postulate, further, either Condition A or Condition B, below. Then we shall have

$$\Psi'(\eta) = \psi(\eta). \tag{6}$$

CONDITION A. $f(x) - \eta$ vanishes for but a finite number of values x_i of x; and each $f'(x_i)$ exists and is not zero.

CONDITION B. (1) There is a constant M such that for small enough $\Delta\eta$

$$|g'(\eta + \theta\Delta\eta)| < M, \qquad |\theta| \leqq 1. \tag{7}$$

(2) The lengths of intervals between consecutive zeros of $f(x) - \eta$ have a positive lower bound b.

(3) $\varphi(x)$ has but a finite number of discontinuities.

(4) When $|x|$ is sufficiently large, $\varphi(x)$ does not change sign, and $|\varphi(x)|$ does not increase with $|x|$.

PROOF. Under Condition A and also under Condition B, because of 2^0, the intersections x_i of $y = f(x)$ with $y = \eta$ are isolated. The curve—see 1^0—cuts $y = \eta$, and is not merely tangent to it. We may suppose, then, that for x_i with even subscripts, $g'(\eta) \leqq 0$, and for odd subscripts $g'(\eta) \geqq 0$. With i even, take v_i so that $x_i < v_i < x_{i+1}$,—this becomes simply $x_i < v_i$ in case x_i is the greatest abscissa of intersection. Set

$$(8) \qquad u_i = \int_{x_i}^{v_i} \varphi(x)dx; \qquad u_{i+1} = \int_{v_i}^{x_{i+1}} \varphi(x)dx.$$

Then from (5), using all values of i involved, odd and even,

$$(9) \qquad \Psi(\eta) = \sum u_i.$$

But, since $\varphi(x)$ is continuous at x_i,

$$(10) \qquad u_i' = \frac{du_i}{d\eta} = \varphi(x_i)\left|\frac{dx_i}{d\eta}\right|.$$

Under Condition A, then, (6) follows from (3), (4), (9), and (10).

Now, under Condition B, we may take $\eta\Delta$ so small that, on account of $1^0, 3^0$, $\varphi(x)$ is continuous when $\eta - \Delta\eta \leqq y \leqq \eta + \Delta\eta$. In (8) and (9) we may now think of η as replaced by any y in $(\eta - \Delta\eta, \eta + \Delta\eta)$ to form terms u_i with derivatives u_i'. Suppose that 4^0 is applicable for $x \geqq x_n - b$. Then, by (7), (10), and 2^0,

$$(11) \qquad \sum_n^\infty |u_i'| < M \sum_n^\infty |\varphi(x_i)| \leqq \frac{M}{b}\left|\int_{x_n - b}^{\infty} \varphi(x)dx\right|.$$

And, since a similar inequality can be set up for terms with $i \leqq -n$, and $\varphi(x)$ is integrable, $\sum u_i'$ converges; and also $\sum u_i$ converges. Thus (6) is also valid under Condition B, by virtue of a theorem* of function theory.

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* Dini, *Grundlagen für eine Theorie der Functionen einer veränderlichen reellen Grösse*, Leipzig, 1892, p. 154.

Porter, *On the differentiation of an infinite series term by term*, ANNALS OF MATHEMATICS, (2), vol. 3 (1901), pp. 19–20.

If (6) fails for not more than a finite number of values of η

$$\Psi(\eta) = \int_{-\infty}^{\eta} \sum \varphi[g(y)]\,|\,g'(y)\,|\,dy; \tag{12}$$

and, indeed, less restrictive conditions may be given.

To show that the conditions of the theorem are not altogether superfluous, let us form a block diagram, and then round off slightly the corners to make it the graph of a single-valued function,—a graph very roughly representing the wave curve of damped vibration. Suppose, then, that rectangles of unit base are placed alternately above and below the X-axis, resting upon this axis, and each with a vertex upon $x^2y^2 = 1$, except near the origin. And suppose that $\varphi(x) = |x|^{-3/2}$ except near the origin. Then $\Psi'(0) = +\infty$; but $\psi(0) = 0$, since at each intersection, $g'(0) = 0$. The "curve" just described may be modified so as to oscillate about an infinity of lines parallel to the X-axis; and thus for an unlimited number of values in a finite interval $\Psi'(y) \neq \psi(y)$.

The application of the theorem to probability is obvious. The $\Psi(\eta)$ and $\Psi'(\eta)$ in (5) and (6) are respectively the cumulative frequency law and the frequency law or "frequency density" at η for the function $Y = f(X)$, when X is subject to $\varphi(x)$. E. g., if $Y = \cos X$, and $\varphi(x) = 1/2e^{-|x|}$,

$$\psi(y) = \frac{e^{-x_0} + e^{-x_1}}{1 - e^{-2\pi}} \frac{1}{\sqrt{1-y^2}}; \tag{13}$$

$x_0 = \text{arc cos } y$, $0 \leqq x_0 < \pi$, $x_1 = \text{arc cos } y$, $\pi \leqq x_1 < 2\pi$, for $|y| < 1$; and $\psi(y) = 0$ for $|y| > 1$.

3. *The Determination of a Function connecting two Frequency Laws.*

THEOREM II. *For $a \leqq x \leqq b$, $\alpha \leqq y \leqq \beta$, where a, b, α, β, are constants, finite or infinite, let $\varphi(x)$ and $\psi(y)$ be positive except possibly for isolated values of x or y; and let these functions have finite integrals. Set*

$$\Phi(\xi) = \int_a^{\xi} \varphi(x)dx; \qquad \Psi(\eta) = \int_{\alpha}^{\eta} \psi(y)dy. \tag{14}$$

Suppose that for some $c \leqq b$,

$$\Psi(\beta) = \Phi(c). \tag{15}$$

Then there exists a single-valued continuous increasing function,

$$Y = F(X), \tag{16}$$

defined in the interval (a, c) *such that for any* ξ *in* (a, c),

$$\Psi(\eta) = \Phi(\xi); \qquad \eta = F(\xi). \tag{17}$$

PROOF. Set

$$u = \Psi(\eta). \tag{18}$$

Then u is a continuous increasing function of η in (α, β). Hence

$$\eta = \Psi^{-1}(u \tag{19}$$

is a continuous increasing function of u in $[\Psi(\alpha), \Psi(\beta)]$, that is, by (15), in $[\Phi(a), \Phi(c)]$. Hence

$$\eta = \Psi^{-1}[\Phi(\xi)] = F(\xi) \tag{20}$$

is a continuous increasing function of ξ in (a, c).

This theorem suggests that although a cause X may be subject to some generally recognized frequency law $\varphi(x)$, its effect Y may be subject to almost any imaginable law $\psi(y)$, in the absence of rather definite knowledge of the functional relation between X and Y. In this connection, not only would $F(X)$ be available, as a possible expression of this unknown relation, but an infinite number of functions $f(X)$ to which Theorem I applies.

4. *Conclusion.* If the conditions of both theorems are satisfied everywhere, then there exists one and only one continuous increasing function $F(X)$ whose frequency law is identical with that of $f(X)$,—the cumulative law $\Psi(\eta)$, indeed, being the same for $F(X)$ and for $f(X)$ if only a finite number of points need to be excepted. Thus $F(X)$ may be looked upon as the chief representative of a whole class of functions $f(X)$ associated with each other through $\varphi(x)$.

THE UNIVERSITY OF TEXAS

ON THE ACCESSIBILITY OF AN ARC FROM ITS COMPLEMENT IN SPACE OF THREE DIMENSIONS

(Extract from a letter to R. L. Moore, dated Jan. 29, 1924)

BY CASIMIR KURATOWSKI

In a paper on continuous curves printed in the BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, July, 1923, you have raised the question, *whether an arc lying in a three-dimensional space is accessible at each of its ends from every point which does not lie on it.*

I shall answer *positively* this question.

Let AB be an arc lying in the space S, X a point of $S-AB$, P a point of AB (not necessarily an end-point). Let H be a plane containing the points X and P.

The point-set G, composed of those points of the arc AB which lie on H, is a closed plane and bounded set. Since the arc AB passes through G, the set G does obviously satisfy a condition (which you gave with Professor Kline in a paper printed in ANNALS OF MATHEMATICS)* which is necessary and sufficient in order that it should be possible to pass an arc through it. Let T be an arc lying on H, containing G but not X. Hence T is accessible at each of its points on the plane H (see, e. g., Schoenflies). Let XP be an arc lying on H and having in common with T only the point P. Hence XP has in common with AB only the point P and thus the theorem is proved.

Obviously the proof holds true in n-dimensional space, $n \geqq 3$.

THE UNIVERSITY OF WARSAW

* The paper here referred to is *On the most general plane closed point-set through which it is possible to pass a simple continuous arc*, ANNALS OF MATHEMATICS, (2), vol. 20 (1919), pp. 218–223. R. L. M.

ON SETS OF THREE CONSECUTIVE INTEGERS WHICH ARE QUADRATIC OR CUBIC RESIDUES OF PRIMES*

BY H. S. VANDIVER

1. *Introduction.* The problem of finding sets of two consecutive integers which are quadratic residues of a prime has been considered by a number of writers† from the point of view of finding integers x and y such that

$$x^2 \equiv y^2 + 1 \pmod{p}$$

p being a prime. I know of no references however, on the problem of finding three consecutive integers which are squares.

As to consecutive integers which are cubic residues, the congruence

$$x^3 \equiv y^3 + 1 \pmod{p}$$

has been studied,‡ but the problem of determining sets of three consecutive integers or sets in arithmetic progression which are cubic residues has apparently not been considered.

In the present note special results on such distribution of quadratic and cubic residues will be obtained.

2. *Three Consecutive Quadratic Residues.* It is known that

$$\frac{x^5 - 1}{x - 1} \equiv 0 \pmod{p}$$

has solutions prime to 5 only when the prime $p \equiv 1 \pmod{5}$.

* Presented to the Society, December 29, 1923. The author was enabled to carry out this investigation through a grant from the Heckscher Foundation for the Advancement of Research, established by August Heckscher at Cornell University.

† See the references in Dickson, *History of the Theory of Numbers*, vol. 2, pp. 282–303 (These are included incidentally in the literature on representation of a number as the sum of four squares.)

‡ See Libri[24], Pellet[128–244], Dickson[199], Cornacchia[217], Mantel[277], Hurwitz[213], Schur[283], of Chapter 26 of vol. 2 of Dickson's *History*.

Under this assumption put $v=x+1/x$; then $(x^5-1)/(x-1)=0$ may be written $v^2+v-1=0$ and $v\equiv(-1\pm a)/2 \pmod p$ where $a^2\equiv 5 \pmod p$. From $v=x+1/x$ it follows that the congruence*

$$\frac{-1\pm a}{2}\equiv x+\frac{1}{x} \pmod p$$

has solutions in x. Hence the discriminant is a square modulo p, or there exists an integer u such that

$$\begin{aligned} 2u^2 &\equiv -5\pm a \pmod p,\\ 2u^2 &\equiv -a^2\pm a \pmod p,\\ -2au^2 &\equiv a^2(a\pm 1) \pmod p \end{aligned} \tag{1}$$

and, using Legendre's quadratic residue symbol we have

$$\left(\frac{a\pm 1}{p}\right)=\left(\frac{-2a}{p}\right). \tag{2}$$

If p is of the form $5n-1$, then the congruence $x^2\equiv 5 \pmod p$ has a solution, but $(x^5-1)/(x-1)\equiv 0 \pmod p$ has no solution. Hence

$$\left(\frac{a\pm 1}{p}\right)=-\left(\frac{-2a}{p}\right); \tag{3}$$

(2) and (3) prove a proposition proposed as a problem in the AMERICAN MATHEMATICAL MONTHLY.†

From (1) we have $4u^2\equiv -10\pm 2a \pmod p$ for $p=5n+1$, but no such congruence holds for $p=5n-1$. Hence we have the following theorem.

THEOREM I. *If* $a^2\equiv 5 \pmod p$, *then*

$$\left(\frac{10+2a}{p}\right)=\left(\frac{10-2a}{p}\right)=\pm 1$$

according as p *is a prime of the form* $5n\pm 1$.

In (2), let p be a prime of the form $40k+11$. Then $(-2/p)=1$, and therefore

* The fraction m/n in a congruence stands for the integer z, where $zn\equiv m \pmod p$.

† Problem 152, vol. 15, 1908, p. 235.

$$\left(\frac{a+1}{p}\right) = \left(\frac{a}{p}\right) = \left(\frac{a-1}{p}\right)$$

and $a+1$, a and $a-1$ are successive integers which are all quadratic residues of p, or all quadratic non-residues. There are two roots of $x^2 \equiv 5 \pmod{p}$, so if $(a/p) = -1$, then $(p-a)/p = 1$, and we have therefore determined three successive integers which are all quadratic residues of p. If $p = 40k+39$, we have by (3)

$$\left(\frac{a+1}{p}\right) = \left(\frac{a-1}{p}\right) = -\left(\frac{-2}{p}\right)\left(\frac{a}{p}\right) = \left(\frac{a}{p}\right)$$

and as before we may select a to be a quadratic residue of p. If $p = 40n+1$, then

$$\left(\frac{a+1}{p}\right) = \left(\frac{a-1}{p}\right) = \left(\frac{a}{p}\right)$$

and these are not quadratic residues of p unless a is a quadratic residue of p, or in other words 5 is a biquadratic residue of p. Since p is of the form $40n+1$, it is not always possible to select a so that it is a quadratic residue. Similar remarks apply to the case $p = 40n+29$. Hence we have the following theorem.

THEOREM II. *If p is of one of the forms $40k+1$, $40k+11$, $40k+29$, $40k+39$, then $a+1$, a, $a-1$ are all quadratic residues of p, where $a^2 \equiv 5 \pmod{p}$ and 5 is a biquadratic residue of p. If p is of one of the two forms $40k+11$, $40k+39$, it is always possible to select a so that $(a/p) = 1$. If p is one of the two forms $40k+1$, $40k+29$, then if one set $a+1$, a, $a-1$, exists then at least one other exists, namely $(p-a)+1$, $p-a$, $(p-a)-1$, having the desired property.*

In case we are testing the existence of sets in the case $p = 40k+1$ or $40k+29$, and p is large, we may avoid the determination of a by the use of the law of biquadratic reciprocity,* which enables us to find the value of $(5/p)_4$.

* Smith, *Report on the theory of numbers*, COLLECTED WORKS, vol. 2, p. 77.

EXAMPLES: The three consecutive integers 3, 4, and 5 are quadratic residues of 11; also 19, 20, and 21 are quadratic residues of 79.

It is evident that we can also set up, by similar methods, theorems regarding integers a, $a+1$, $a+2$, modulo p, where each of these integers have prescribed quadratic characters.

3. *Cubic Residues.* We shall now discuss cubic residues. We shall first prove the following lemma.

LEMMA. *If p is an odd prime >3, and*

$$y^3+ay+b \equiv 0 \pmod{p} \tag{4}$$

where a, b, and y are integers, and also k is an integer not congruent to 0 (mod p) such that

$$R = \frac{b^2}{4}+\frac{a^3}{27} \equiv k^2 \pmod{p}$$

then

$$\left(-\frac{b}{2}+k\right) \text{ and } \left(-\frac{b}{2}-k\right)$$

are cubic residues modulo p.

To show this we follow a procedure analogous to that employed in Cardan's solution of the cubic. Let $y=u+v$; then (4) gives $3uv=-a$, $u^3+v^3=-b$. Assuming that y is an integer, then $u^2+2uv+v^2$ is an integer. Assuming $p>3$, then $3uv=-a$ shows that uv is congruent to an integer modulo p, hence u^2+v^2 has the same property, as well as u^2+uv+v^2. Also since $(u^3-v^3)^2=4R$ the same is true of u^3-v^3. Now $(u-v)(u^2+uv+v^2)=u^3-v^3$ and $k\not\equiv 0 \pmod{p}$, whence $u-v$ is congruent to an integer modulo p. Since $u+v$ is also, we have both u and v congruent to integers modulo p. Since $u^3=(-b/2+k)$ and $v^3=(-b/2-k)$ the lemma is proved.

We now apply the above result to the function

$$f(x) = x^3-3\lambda x-m\lambda \tag{5}$$

where λ is a prime,

$$4\lambda = m^2+27n^2, \quad m \equiv 1 \pmod 3.$$

We also have

$$f(x) = (x-\eta_1)(x-\eta_2)(x-\eta_3)$$

where

$$\eta_1 = \alpha + \alpha^{r^3} + \alpha^{r^6} + \cdots + \alpha^{r^{\lambda-4}},$$

$$\eta_2 = \alpha^r + \alpha^{r^4} + \alpha^{r^7} + \cdots + \alpha^{r^{\lambda-3}},$$

$$\eta_3 = \alpha^{r^2} + \alpha^{r^5} + \alpha^{r^8} + \cdots + \alpha^{r^{\lambda-2}},$$

$\alpha^\lambda = 1$, $\alpha \neq 1$, r a primitive root of λ. By a theorem of Kummer* we know that the congruence $f(x) \equiv 0 \pmod p$ always has solutions if $p^{(\lambda-1)/3} \equiv 1 \pmod \lambda$. Identifying (5) with (4) we have, since $a = -3\lambda$, $b = -m\lambda$,

$$R = \frac{-27n^2\lambda^2}{4}.$$

Now if p is a prime congruent to 1 or 7, modulo 12, then $(-3/p) = 1$, so that an integer k exists so that $k^2 \equiv R \pmod p$. Put $j^2 \equiv -3 \pmod p$; then we may write

$$k \equiv \frac{j^2 n}{2} \pmod p,$$

so that, from the lemma,

$$\frac{\lambda m}{2} + \frac{n\lambda}{2} j^3, \qquad \frac{\lambda m}{2} - \frac{n\lambda}{2} j^3$$

are cubic residues of p.

Hence, if both $4\lambda m$ and $4\lambda n$ are cubic residues, so that $4\lambda m \equiv g^3$ and $4\lambda n j^3 \equiv h^3 \pmod p$, then

$$\left(\frac{g}{2}\right)^3 + \left(\frac{h}{2}\right)^3, \quad \left(\frac{g}{2}\right)^3, \quad \left(\frac{g}{2}\right)^3 - \left(\frac{h}{2}\right)^3$$

are all cubic residues and if $(h/2)f \equiv 1 \pmod p$ then, modulo p,

* CRELLE, vol. 30 (1846), pp. 107–116.

$$\left(\frac{fg}{2}\right)^3+1, \quad \left(\frac{fg}{2}\right)^3, \quad \left(\frac{fg}{2}\right)^3-1$$

give a set of consecutive integers which are cubic residues. If either $4\lambda m$ or $4\lambda n$ is a cubic residue then we may find three cubic residues in arithmetic progression. Noting that if s is a cubic residue of p then $(-s)$ is also, and

$$\frac{n\lambda}{2}j^3-\frac{\lambda m}{2}$$

is a cubic residue, the result follows easily. We may then state the following theorem.

THEOREM. *If λ is a prime,*

$$\lambda=\frac{m^2+27n^2}{4},\ m\equiv 1 \pmod 3,\ p\equiv 1 \text{ or } 7 \pmod{12},$$

$p^{(\lambda-1)/3}\equiv 1 \pmod\lambda$ *and if $4\lambda m$ or $4\lambda n$ is a cubic residue of p then it is possible to find three cubic residues of p in arithmetic progression. If both $4\lambda m$ and $4\lambda n$ are cubic residues, it is possible to find three consecutive integers which are cubic residues of p.*

EXAMPLE. Let $p=37$, $\lambda=19$, whence $m=7$, $n=1$, $j=16$. Here we find $4\lambda m=14$ is a cubic residue of 37 and 36, 11, 23 are cubic residues of 37 in arithmetic progression, with common difference 12.

The methods employed above apply to other similar problems, but I have not been able to obtain any theorems which are very general. In particular owing to the use of the equations $\varphi(z)=(z-\eta_1)(z-\eta_2)\cdots(z-\eta_e)$, where $\eta_1, \eta_2, \cdots, \eta_e$ are the cyclotomic periods, $ef=n-1$, of the equation $x^n=1$, in the algebraic solution of this equation, and the results of Kummer already cited on the integral divisors of $\varphi(z)$, it is possible to obtain a variety of results, but they all seem to be rather special in character, as are the theorems of this paper.

CORNELL UNIVERSITY

NORMAL CONGRUENCES OF CURVES IN RIEMANN SPACE*

BY HARRY LEVY

The purpose of this note is to determine necessary and sufficient conditions that a congruence of curves in a Riemann space of n dimensions be normal to a family of k—dimensional hypersurfaces. We proceed first to find certain necessary conditions.

If the hypersurfaces, which we denote for brevity by V_k, are defined by the equations

$$f_i(x^1 x^2 \cdots x^n) = c_i, \quad (i = 1, 2, \cdots, n-k), \tag{1}$$

where the c's are constants, any one of the $n-k$ vectors, $\partial f_i/\partial x^n$, $(i = 1, 2, \cdots, n-k)$ will be orthogonal to the family. Hence, if a congruence of curves is normal to a family V_k, there are $n-k$ linearly independent congruences normal to the same family. Define these congruences by the $n-k$ systems of equations

$$\frac{dx^1}{\lambda_h|^1} = \frac{dx^2}{\lambda_h|^2} = \cdots = \frac{dx^n}{\lambda_h|^n}, \quad (h = 1, 2, 3, \cdots, n-k). \tag{2}$$

There will be no loss of generality in assuming that the congruences are mutually orthogonal; if we denote the fundamental tensor of the space by g_{ij}, that is, if the linear element is given by the positive definite form

$$ds^2 = g_{ij} dx^i dx^j, \dagger \tag{3}$$

then we may write

$$g_{rs} \lambda_h|^r \lambda_j|^s = \delta_{hj}, \tag{4}$$

* Presented to the Society, April 28, 1923.

† An index repeated, once a subscript and once a superscript, is summed from 1 to n.

for $h, j = 1, 2, \cdots, n-k$, where δ_{hj} is Kronecker's delta,

$$\delta_{hj} = \begin{cases} 0, & h \neq j, \\ 1, & h = j. \end{cases} \tag{5}$$

Take any other k congruences, defined by (2) for $h = n-k+1, \cdots, n$, which are mutually orthogonal and orthogonal to the first group. Then (4) will hold for all values of h and j from 1 to n.

The coefficients of rotation, γ_{hij}, of this orthogonal ennuple will be defined by

$$\gamma_{hij} = \lambda_{h|r,s}\lambda_i|^r \lambda_j|^s, \tag{6}$$

where $\lambda_{h|r}$ for $r = 1, 2, \cdots, n$ are the covariant components of the vector $\lambda_{h|}$, that is $\lambda_{h|r} = g_{rs}\lambda_h|^s$, and $\lambda_{h|r,s}$ is the covariant derivative of $\lambda_{h|r}$,

$$\lambda_{h|r,s} = \frac{\partial \lambda_{h|r}}{\partial x^s} - \lambda_{h|t}\Gamma_{rs}|^t. \tag{7}$$

The functions $\Gamma_{rs}|^t$ are the Christoffel symbols of the second kind.

Ricci has shown* that we may solve (6) for $\lambda_{h|r,s}$, and that

$$\lambda_{h|r,s} = \sum_{i,j=1}^{n} \gamma_{hij}\lambda_{i|r}\lambda_{j|s}. \tag{8}$$

If congruences $\lambda_{1|}, \lambda_{2|}, \cdots, \lambda_{n-k|}$ have a family of k-dimensional hypersurfaces as orthogonal trajectories, every direction orthogonal to these $n-k$ congruences must belong to the hypersurface. Hence the equations†

$$X_j(f) \equiv \lambda_j|^r \frac{\partial f}{\partial x^r} = 0, \qquad (j = n-k+1, \cdots, n), \tag{9}$$

must have $n-k$ independent solutions. A necessary and

* See *Méthodes de calcul différentiel absolu*, Ricci and Levi-Civita, MATHEMATISCHE ANNALEN, vol. 54 (1901), p. 148. As will be seen later, the result of this paper coincides with that of Ricci for the case $k = n-1$.

† Regarding the notation here used, see Goursat, *Les Equations Différentielles du Premier Ordre.*

sufficient condition that (9) be a complete system admitting $n-k$ independent solutions is that $(X_iX_j-X_jX_i)f$ be a linear function of the expressions $X_m(f)$. Now

$$X_iX_j(f) = \lambda_i|^r \frac{\partial}{\partial x^r}\left(\lambda_j|^s \frac{\partial f}{\partial x^s}\right) = f_{rs}\lambda_i|^r \lambda_j|^s + f_s\lambda_j|^s{}_{,r}\lambda_i|^r$$

where

$$\lambda_j|^s{}_{,r} = g^{st}\lambda_{j|t,r},$$

where f_r is $\partial f/\partial x^r$, and where f_{rs} is the covariant derivative of f_r. Making use of (8) to eliminate $\lambda_j|^s{}_{,r}$, we find, by (4) and (9),

$$X_iX_j(f) = f_{rs}\lambda_i|^r\lambda_j|^s + \sum_{l=n-k+1}^{n}\gamma_{jli}X_l + \sum_{h=1}^{n-k}\gamma_{jhi}f_s\gamma_h|^s.$$

Since f_{rs} is symmetric, we have

$$(X_iX_j-X_jX_i)f = \sum_{l=n-k+1}^{n}(\gamma_{jli}-\gamma_{ilj})X_l + \sum_{h=1}^{n-k}(\gamma_{jhi}-\gamma_{ihj})f_s\lambda_h|^s.$$

Hence we may write

$$\sum_{h=1}^{n-k}(\gamma_{jhi}-\gamma_{ihj})f_s\lambda_h|^s = 0.$$

All the coefficients of $f_s\lambda_h|^s$ must vanish, otherwise we would have another relation between the derivatives of f. Consequently, necessary conditions are given by the equations

$$\gamma_{jhi}-\gamma_{ihj} = 0, \qquad (h = 1, 2, \cdots, n-k;\ i, j = n-k+1, \cdots, n). \tag{10}$$

These conditions are also sufficient, hence we have the following theorem.

THEOREM. *The necessary and sufficient condition that congruences $\lambda_1|, \lambda_2|, \cdots, \lambda_{n-k}|$ have a family of k-dimensional hypersurfaces as orthogonal trajectories is that equations* (10) *be satisfied.* When $k = n-1$ this coincides with Ricci's results.*

* In a paper presented to the International Mathematical Congress in August, 1924, Ricci, unaware of my work, also obtained this general result using essentially the preceding method.

Suppose every set of $n-k$ congruences of the orthogonal ennuple is normal to a family V_k. Then equations (10) are satisfied when h, i and j take on any distinct values. But Ricci has shown that $\gamma_{hij} = -\gamma_{ihj}$; combining, we find $\gamma_{hij} = 0$ for h, i, j distinct, and this is a sufficient condition that all the congruences be normal. Consequently, *if every set of $n-k$ congruences ($k>1$) of an orthogonal ennuple has a family of k dimensional hypersurfaces as orthogonal trajectories, then all of the congruences are normal.*

PRINCETON UNIVERSITY

LIMITS FOR ACTUAL DOUBLE POINTS OF SPACE CURVES*

BY T. R. HOLLCROFT

1. *Introduction.* Noether† has proved that space curves of maximum genus on non-singular surfaces always exist. In the first part of this paper, such curves will be assumed to exist also when any number of actual double points are added, subject only to the fact that the genus can not be negative for a proper curve and to certain other limitations. The purpose of this paper is to ascertain and define these limitations when all of the actual double points of the space curve are cusps or when all are nodes, and to discuss the existence of space curves with any number of actual double points up to and including the maximum.

From Noether's formula for the maximum genus π_μ of a curve of order n on a non-singular surface of order μ, is obtained the theorem: The minimum number of apparent double points h_μ for a space curve of order n cut out by

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* Presented to the Society, February 24, 1923.

† M. Noether, *Zur Grundlegung der Theorie der Algebraischen Raumkurven*, ABHANDLUNGEN DER PREUSSISCHEN AKADEMIE DER WISSENSCHAFTEN, 1882, Section 6.

a surface of order ν on a non-singular surface of order μ, if $\mu\nu \geqq n > \mu(\nu-1)$, $\mu \leqq \nu$, is

$$h_\mu = \frac{1}{2}(2n-\mu\nu)(\mu-1)(\nu-1).$$

2. *Limits Obtained by Projection.* From an arbitrary point, the apparent double points of a space curve project into plane nodes, but from a point on the developable, from a point on the nodal curve of the developable, or from a triple point of this curve, one, two or three apparent double points respectively are projected into cusps on the plane curve of projection. For $n>3$ the nodal curve always exists and for $n \geqq 8$ it always has triple points.

Let C_n represent a space curve of order n and maximum genus π_μ with h_μ apparent double points and lying on a non-singular surface of order μ. Let C_n be projected into a plane curve $\bar{C}_n$, for $n \leqq 7$, from any point of the nodal curve of the developable surface. Resulting from the projection of the apparent double points of C_n, the plane curve $\bar{C}_n$ will have two cusps and

$$\delta_1 = h_\mu - 2 = \frac{1}{2}(2n-\mu\nu)(\mu-1)(\nu-1) - 2$$

nodes. For $n \geqq 8$ the projection will be made from a triple point of the nodal curve and the plane curve will then have three cusps and $\delta_1 = h_\mu - 3$ nodes.

The maximum number of cusps that may be added to the singularities of this plane curve $\bar{C}_n$ is the number of cusps of $\bar{C}_n$ that can result from the projection of cusps of C_n, that is, it gives an upper limit for the number of cusps of the space curve. The solution of this problem for a plane curve is given in a former paper.* Use the above values of δ_1 in the limits of δ_1 given in the solution of this problem, denote the maximum number of cusps of C_n by β_μ and we have the result:

* T. R. Hollcroft, *Singularities of curves of given order*, this BULLETIN, vol. 29 (1923), Problem 1, pp. 409–10.

The maximum number of cusps β_μ for C_n as defined above, provided that $\mu\nu \geqq n > \mu(\nu-1)$, is as follows:

For $n \leqq 7$,

$$\beta_\mu = \frac{1}{2}(n-1)(n-2) - h_\mu; \tag{1}$$

For $n \geqq 8$,

$$\text{If } h_\mu \geqq \frac{1}{2}n(n-10) + 3 + 2\sqrt{4n+13},$$

$$[\beta_\mu]^* \leqq \frac{3}{8}[n(n-2) - 2(h_\mu+1)]; \tag{2}$$

$$\text{If } \frac{1}{2}n(n-10)+3+2\sqrt{4n+13} > h_\mu > \frac{1}{2}n(n-13) + 14 - \sqrt{16n-23},$$

$$[\beta_\mu] \geqq \frac{1}{6}[2n(n-1) - 4h_\mu - 15 - \sqrt{4n(n-9) - 8h_\mu + 105}\,]; \tag{3}$$

$$\text{If } h_\mu \leqq \frac{1}{2}n(n-13) + 14 - \sqrt{16n-23},$$

$$[\beta_\mu] \leqq \frac{1}{4}[(n-3)(n+6) - 2(h_\mu+3)]. \tag{4}$$

An exception to (1) occurs when $\mu = 2$ since a C_7 on a quadric can have but five cusps.

Since $\mu\nu \geqq n > \mu(\nu-1)$, in (2)

$$h_\mu \geqq \frac{1}{2}n(n-10) + 3 + 2\sqrt{4n+13}$$

only for $\mu + \nu \leqq 8$; and in (3)

$$h_\mu \geqq \frac{1}{2}n(n-13) + 14 - \sqrt{16n-23}$$

only for $\mu + \nu \leqq 14$.

If in the formula for h_μ we let $\mu = 2$ and $\nu = n/2$ or

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* The symbol $[x]$ followed by $\leqq$ ($\geqq$) means the largest (one greater than the largest if not equal) integer contained in the expression on the right.

$(n+1)/2$ we obtain $h_2 = \frac{1}{4}n(n-2)$ or $\frac{1}{4}(n-1)^2$ for n even or odd respectively. The inequality

$$[h_2] \geqq \frac{1}{4}n(n-2)$$

defines the smallest number of apparent double points for a proper space curve of order n.* Then when $\mu = 2$, C_n can have the largest number of actual double points possible for a given order n.

Using the above value of h_2 in the preceding limits, we can determine the values of n for which the several limits of δ_1 hold and thus obtain the following formulas.

For $n \leqq 6$,

$$[\beta_2] \leq \frac{1}{4}(n-2)^2;$$

For $7 \leqq n \leqq 11$,

$$[\beta_2] \leqq \frac{3}{16}(n^2-2n-4);$$

For $12 \leqq n \leqq 25$,

$$[\beta_2] \geqq \frac{1}{6}(n^2-15-\sqrt{2n(n-12)+105});$$

For $n \geqq 26$,

$$[\beta_2] \leq \frac{1}{8}(n^2+8n-44).$$

With the exception of the limit for $n \leqq 6$, these are only upper bounds for β_2 and can not be attained, as will be shown in the two following sections.

Since every rational plane curve of order $n \geqq 4$ may have three or more cusps, the above considerations do not limit the nodes of a space curve.

3. *Limits given by* $\gamma \geqq 0$. Certain limits for β arise from the fact that none of the characteristic numbers of a

* G. Halphen, *Sur quelques propriétés des courbes gauches algébriques*, BULLETIN DE LA SOCIÉTÉ DE FRANCE, vol. 2, p. 42.

proper space curve can be negative, and in some cases these limits are more restrictive than those derived above by projection. Of these, the lowest limit for β is given by γ, the number of cusps of the nodal curve. For $\gamma \geqq 0$, there results the inequality

$$[\beta] \leqq \frac{1}{6}[r(n-6)+4(n-\omega-H)-2v],$$

wherein r is the rank, v the number of linear inflections, ω the number of double tangents and H the number of nodes of a space curve. When $\omega = v = 0$, as may happen for curves of any order, the space curve can have a larger number of cusps. There results

$$[\beta] \leqq \frac{1}{6}[r(n-6)+4(n-H)].$$

This serves as an upper bound to β for all space curves.

Space curves that lie on a quadric have $[h] \geqq n(n-2)/4$, so that

$$[r] \leqq \frac{n^2}{2}-3\beta-2H.$$

Substitute this limit for r in the foregoing inequality derived from $\gamma \geqq 0$ and we obtain

$$[\beta_2] \leqq \frac{1}{6}[n(n-2)-4H].$$

Since, for a maximum β, $H = 0$ or 1, the limit becomes

$$[\beta_2] \leqq \frac{1}{6}n(n-2).$$

If, when $n(n-2)$ is divided by 6, the remainder is greater than or equal 4, there may be one node. This occurs only when n is of the form $n \equiv 1$, mod 6.

This formula gives as small a limit for β_2 for $7 \leqq n \leqq 26$ as those obtained by projection, or a smaller limit.

For curves on cubic surfaces, we find that the inequality $\gamma \geqq 0$ leads to the limit

$$[\beta_3] \leqq \frac{1}{9}(n-2)(n+3).$$

For $n \leqq 8$, all the actual double points may be cusps. The above limit is more restrictive than those derived by projection for $12 \leqq n \leqq 17$ and gives the same value of β_3 for $n = 9, 10, 11$.

For curves on quartic surfaces, there are no values of n for which the limit derived from $\gamma \geqq 0$ gives a smaller limit for β_4 than those given by the projection formulas.

If in the above inequality for β in terms of n and H, we let $\beta = 0$ instead of H, we obtain

$$[H_3] \leqq \frac{1}{4} n(n-2).$$

This limit, however, is always greater than the maximum genus of C_n on a quadric, so the fact that γ cannot be negative does not restrict the number of nodes when $\beta = 0$. The same is true for curves on surfaces of higher order than two.

4. *Limits Caused by Adjoint Curve.* As was shown by Noether,* the necessary and sufficient condition that a non-composite plane curve be the projection of a space curve is that the double points of the plane curve which are projections of the apparent double points of the space curve lie on a plane curve of order $n-i-3$, where $i \leqq \mu+\nu-4$ if $\mu\nu = n$ or $i \leqq \mu+\nu-5$ if $\mu\nu > n > \mu(\nu-1)$. This curve of order $n-i-3$ is called an adjoint curve. There are adjoint curves for all values of i less than or equal to the given limits, but since the one for the largest value of i is of least order and thus imposes the greatest number of conditions on C_n, only that one will be considered here.

First consider curves that are complete intersections of the two surfaces of orders μ and ν, that is, let $\mu\nu = n$. For such curves the greatest value of i is $\mu+\nu-4$ and the order of the adjoint curve of least order is $(\mu-1)(\nu-1)$.

* M. Noether, loc. cit., § 4, Theorem II.

The number of conditions necessary to determine this adjoint curve is $\frac{1}{2}(\mu-1)(\nu-1)[(\mu-1)(\nu-1)+3]$. Subtracting this from $h_\mu = \frac{1}{2}\mu\nu(\mu-1)(\nu-1)$, we find that there are $(\mu-1)(\nu-1)(\mu+\nu-4)/2$ double points of $\overline{C}_n$ through which the adjoint curve passes gratuitously, and therefore that among the h_μ double points of $\overline{C}_n$ there exist $\frac{1}{2}(\mu-1)(\nu-1)(\mu+\nu-4)$ linear relations.

For a sufficiently large n, these conditions on $\overline{C}_n$ lead to a limit to the number of cusps (and, for a still larger n, to the number of nodes) of C_n. The projections of the h_μ apparent double points of C_n, since they may be projected into $h_\mu-3$ nodes and 3 cusps of $\overline{C}_n$, account at most for $h_\mu+3$ conditions on $\overline{C}_n$. Then the number of degrees of freedom of $\overline{C}_n$ is

$$\frac{1}{2}n(n+3)-h_\mu-3-\frac{1}{2}(\mu-1)(\nu-1)(\mu+\nu-4)$$
$$=\frac{1}{2}[6n-2+(\mu+\nu)(\mu+\nu-5)],$$

all of which may be absorbed by assigning cusps or nodes to $\overline{C}_n$. Lefschetz* has proved that a node accounts for one and a cusp for two invariants, and that for the genus $p \geqq 2$, no more than $\frac{1}{2}n(n+3)-8$ of the $\frac{1}{2}n(n+3)$ conditions necessary to determine a plane curve may be invariants. For the curves $\overline{C}_n$ involved here, more than eight metric conditions are given in each case by the linear relations that exist among the double points of $\overline{C}_n$ which are projections of the apparent double points of C_n, so that all the remaining degrees of freedom of $\overline{C}_n$ may be absorbed by assigning nodes or cusps to $\overline{C}_n$.

An upper limit to the number of cusps of C_n is, therefore,

$$[\beta\mu] \leqq \frac{1}{4}[2(3n-1)+(\mu+\nu)(\mu+\nu-5)],$$

and an upper limit to the number of nodes is

$$H_\mu = 3n-1+\frac{1}{2}(\mu+\nu)(\mu+\nu-5).$$

* S. Lefschetz, *On the existence of loci with given singularities*, TRANSACTIONS OF THIS SOCIETY, vol. 14 (1913), pp. 23–41.

When C_n is not a complete intersection, for curves with the minimum number of apparent double points, $\mu\nu > n > \mu(\nu-1)$, $i = \mu+\nu-5$ and the number of linear conditions among the h_μ double points of $\bar{C}_n$ is, for $\mu \leqq \nu$,

$$\frac{1}{6}[i(i+1)(i+2)-(i-\mu+1)(i-\mu+2)(i-\mu+3)]$$

for $\mu \leqq 4$, and

$$\frac{1}{6}[i(i+1)(i+2)-(i-\mu+1)(i-\mu+2)(i-\mu+3) \\ -(i-\nu+1)(i-\nu+2)(i-\nu+3)],$$

for $\mu \geqq 5$.

An upper limit to the cusps of C_n is found as before by subtracting the sum of each of the foregoing expressions and $h_\mu+3$ from $\frac{1}{2}n(n+3)$, which gives, respectively,

$$[\beta\mu] \leqq \frac{1}{4}[(n-1)(n+2)+(\mu+\nu-\mu\nu)(2n+\mu+\nu-\mu\nu-7) \\ -\frac{1}{3}(\mu-2)(\mu-3)(\mu-4)]$$

for $\mu \leqq 4$, and for $\mu \geqq 5$,

$$[\beta\mu] \leqq \frac{1}{4}[(n-1)(n+2)+(\mu+\nu-\mu\nu)(2n+\mu+\nu-\mu\nu-7)].$$

Since the additional terms in the first limit by which it differs from the second vanish identically for values of μ within the limit $\mu \leqq 4$ for which the formula holds, the second limit for $\beta\mu$ holds for all values of μ, $\mu \leqq \nu$ and $\mu\nu > n > \mu(\nu-1)$.

Also as before, the upper limit to the number of nodes of C_n for $\mu \leqq \nu$ and $\mu\nu > n > \mu(\nu-1)$ becomes

$$H_\mu = \frac{1}{2}[(n-1)(n+2)+(\mu+\nu-\mu\nu)(2n+\mu+\nu-\mu\nu-7)].$$

These limits for β_μ are more restrictive than projection limit (2) for $\mu+\nu \geqq 9$, but since limit (2) holds only for $\mu+\nu \leqq 8$, it is not superseded by this new limit.

The new limits are more restrictive than projection limit (3) for $\mu+\nu\geqq 9$. Since projection limit (3) holds for $\mu+\nu\leqq 14$, it is superseded by the new limits for all values of μ and ν such that $\mu+\nu\geqq 9$. Projection limit (4) is entirely superseded by the new limits for all values of μ and ν.

5. *Summary of Limits.* Finally, for any proper space curve of order n which is formed by the intersection of a surface of order ν with a non-singular surface of order μ such that $\mu\nu\geqq n>\mu(\nu-1)$, there exist the following upper limits to the number of cusps it may possess:

When $n\leqq 7$ ($n\leqq 6$ for $\mu=2$),

$$\beta_\mu = \frac{1}{2}(n-1)(n-2)-h_\mu;$$

When $n\geqq 8$ and $\mu+\nu\leqq 8$,

For $h_\mu \geqq \frac{1}{2}n(n-10)+3+2\sqrt{4n+13}$

$$[\beta_\mu] \leqq \frac{3}{8}[n(n-2)-2(h_\mu+1)];$$

For $\frac{1}{2}n(n-10)+3+2\sqrt{4n+13} > h_\mu > \frac{1}{2}n(n-13)+14-\sqrt{16n-23}$

$$[\beta_\mu] \geqq \frac{1}{6}\left[2n(n-1)-4h_\mu-15-\sqrt{4n(n-9)-8h_\mu+105}\right];$$

When $\mu+\nu\geqq 9$,

If $\mu\nu = n$,

$$[\beta_\mu] \leqq \frac{1}{4}[(\mu+\nu)(\mu+\nu-5)+2(3n-1)];$$

If $\mu\nu > n > \mu(\nu-1)$,

$$[\beta_\mu] \leqq \frac{1}{4}[(n-1)(n+2)+(\mu+\nu-\mu\nu)(2n+\mu+\nu-\mu\nu-7)].$$

In the limits derived from the properties of the adjoint curve, if the equality sign holds, the curve C_n with the maximum number of cusps can have no nodes. If the inequality sign holds, it may have one, but only one.

Under the same general conditions as above, when $\beta_\mu = 0$ the following limits exist for the number of nodes H_μ:

When $\mu\nu = n$, $\mu \leqq \nu$,

For $\mu = 2$, $\nu \geqq 15$; $\mu = 3$, $\nu \geqq 11$; all values of μ such that $4 \leqq \mu \leqq 7$ and $\mu + \nu \geqq 13$; all values of $\nu \geqq \mu > 7$;

$$H_\mu = 3n - 1 + \frac{1}{2}(\mu+\nu)(\mu+\nu-5);$$

When $\mu\nu > n > \mu(\nu-1)$, $\mu \leqq \nu$,

For $\mu = 2$, $\nu \geqq 16$; $\mu = 3$, $\nu \geqq 13$; $\mu = 4$, $\nu \geqq 11$ if ν is of form $(n+1)/4$ or $(n+2)/4$, or $\nu \geqq 12$ if ν is of form $(n+3)/4$; $\mu = 5, 6, 7$, $\nu = (n+c)/\mu$, if $c = 1$, $\mu + \nu \geqq 14$ and if $c > 1$, $\mu + \nu \geqq 15$; and for all values of $\nu \geqq \mu > 7$;

$$H_\mu = \frac{1}{2}[(n-1)(n+2) + (\mu+\nu-\mu\nu)(2n+\mu+\nu-\mu\nu-7)].$$

For $\mu = 2$ and n even, the adjoint curve is of order $\frac{1}{2}(n-2)$ and $\frac{1}{8}(n-2)(n-4)$ conditions are imposed on the $\frac{1}{4}n(n-2)$ double points of $\bar{C}_n$. For $7 \leqq n \leqq 26$ the limit

$$[\beta_2] \leqq \frac{1}{6}n(n-2)$$

is lower than that given by the projection limits. The number of conditions imposed by these cusps and nodes on $\bar{C}_n$ together with the linear relations existing among them amount to $(82n - 5n^2 - 96)/24$. This is negative for $n \geqq 16$, so that for $n \geqq 16$

$$[\beta_2] \leqq \frac{1}{6}n(n-2) - \frac{1}{48}(5n^2 - 82n + 96) = \frac{1}{16}(n^2 + 22n - 32).$$

The limit for n odd is found in a similar way. These limits can also be obtained by substituting $\mu = 2$ and $\nu = n/2$ or $(n+1)/2$ respectively in the general limits for β_μ.

For $\mu = 3$ and $12 \leqq n \leqq 17$ the limit

$$[\beta_3] \leqq \frac{1}{9}(n-2)(n+3)$$

is lower than that given by the projection limits. By taking $\mu = 3$ and $\nu = n/3$, $(n+1)/3$, $(n+2)/3$, respectively, and following each through as for $\mu = 2$, we find that within these limits for n the above limit is lower also than the limits derived from the conditions imposed on $\bar{C}_n$ by the adjoint curve.

For $\mu = 3$, the following limits exist.

For $n \leqq 7$,

$$[\beta_3] \geqq \frac{1}{6}(n-1)(n-2);$$

For $8 \leqq n \leqq 11$,

$$[\beta_3] \leqq \frac{1}{8}(n^2-9);$$

For $12 \leqq n \leqq 17$,

$$[\beta_3] \leqq \frac{1}{9}(n-2)(n+3);$$

For $n \geqq 18$,

If $\nu = \dfrac{n}{3}$, $\quad [\beta_3] \leqq \dfrac{1}{36}(n^2+57n-72);$

If $\nu = \dfrac{n+1}{3}$, $[\beta_3] \leqq \dfrac{1}{36}(n^2+65n-116);$

If $\nu = \dfrac{n+2}{3}$, $[\beta_3] \leqq \dfrac{1}{36}(n^2+61n-98).$

For $\nu = \dfrac{n}{3}$ and $n \geqq 33$,

$$H_3 = \frac{1}{18}(n^2+57n-72);$$

For $\nu = \dfrac{n+1}{3}$ and $n \geqq 38$,

$$H_3 = \frac{1}{18}(n^2+65n-116);$$

For $\nu = \dfrac{n+2}{3}$ and $n \geqq 37$

$$H_3 = \frac{1}{18}(n^2+61n-98).$$

When $\mu = 2$, in view of the fact that space curves with the least number of apparent double points for a given n lie on a quadric, this theorem results:

Every proper space curve with the maximum number of cusps for a given order n lies on a quadric and the maximum for each order is given by the following limits.

(I) For $n \leqq 6$,

$$[\beta_2] \leqq \frac{1}{4}(n-2)^2;$$

(II) For $7 \leqq n \leqq 15$,

$$[\beta_2] \leqq \frac{1}{6}n(n-2);$$

(III) For n even and $n \geqq 16$,

$$[\beta_2] \leqq \frac{1}{16}(n^2+22n-32);$$

(IV) For n odd and $n \geqq 17$,

$$[\beta_2] \leqq \frac{1}{16}(n^2+24n-41).$$

For $n = 7$ the curve may be rational, that is, it may have one node in addition to the five cusps. For $n > 7$, space curves with the maximum number of cusps for a given n are irrational. For some values of n they may have one node, for others, none.

With regard to nodes, a similar statement may be made, viz., every proper space curve with the maximum number of nodes for a given order n lies on a quadric, and the

maximum for each order is given by the following limits (no limit except genus for $n < 30$).

For n even and $n \geqq 30$,

$$H_2 = \frac{1}{8}(n^2 + 22n - 32);$$

For n odd and $n \geqq 31$,

$$H_2 = \frac{1}{8}(n^2 + 24n - 41).$$

6. *Existence of Curves.* Since for $n \geqq 4$, plane curves with three cusps and any number of nodes up to the maximum allowed by the genus exist, and since any two surfaces may have as many ordinary contacts as the apparent genus of their curve of intersection, there are no further questions as to the existence of space curves with any number of nodes up to and including the maximum, except those relating to the adjoint curve and they will be considered in the discussion of the existence of space curves with cusps.

The fact proved by Noether that space curves with only apparent double points, of maximum genus and lying on non-singular surfaces always exist has been mentioned above. The addition of cusps requires that the two surfaces containing the C_n have that number of stationary contacts. So far as the surfaces are concerned, they may have as many stationary contacts as the apparent genus of C_n. In regard to C_n itself, the addition of cusps reduces all the other characteristic numbers, some very rapidly, notably γ, which serves as an effective limit on β. Finally, when the maximum β has been so chosen that all these characteristic numbers are non-negative (with the single exception of the genus of the nodal curve, which may be negative, since the nodal curve may be composite when the cuspidal curve is not) there appears to be nothing more appertaining only to the space curve itself to interfere with its existence.

If, however, we attempt to project C_n upon a plane when β is limited only as above, we find that for n greater than

certain limits, no proper plane curve exists into which C_n may be projected and therefore that C_n itself does not exist as a proper space curve. In order that the plane curve of projection exist, two conditions must be satisfied—first, the number of cusps and nodes assigned to it must not exceed the maximum number a plane curve of that order can possess, and second, the nodes and cusps of the plane curve which are projections of the apparent double points of the space curve, must lie on the adjoint curve.

The existence of plane curves with any number of cusps up to and including the maximum has been proved by both Lefschetz* and Coolidge† for $p \leq p_0$, where p_0 is the genus associated with a given n on the assumption that the bitangents and inflections are both as near zero as possible. Although lacking a formal proof as yet, the existence of curves for $p > p_0$ with any number of cusps up to and including the maximum is practically assured.

In all the preceding, we have used the condition that the double points of the plane curve of projection, which are projections of the apparent double points of the space curve, must lie on an adjoint curve only as the necessary condition that the plane curve be the projection of the space curve. But Noether‡ and Valentiner§ have both proved that this condition is also sufficient—that such a plane curve is always the projection of a space curve.

Therefore the existence of space curves with cusps up to and including the maximum as given by the preceding limits is established, if plane curves for $p > p_0$ with any number of cusps up to and including the maximum exist.

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* S. Lefschetz, loc. cit., pp. 33–39.

† J. L. Coolidge, *On the existence of curves with assigned singularities* this BULLETIN, vol. 28 (1922), pp. 451–55.

‡ M. Noether, loc. cit., § 3, Theorem I.

§ H. Valentiner, *Bidrag til Rumcurvenes Theori*, Inaugural Dissertation, Kopenhagen, 1881.

SOME MEAN-VALUE THEOREMS CONNECTED WITH COTES'S METHOD OF MECHANICAL QUADRATURE

BY D. V. WIDDER

1. *Introduction.* It is the purpose of this paper to prove several mean-value theorems whose importance in the problem of mechanical quadrature will appear later.

2. *Extension of a Theorem of Birkhoff.* The following theorem has been proved by G. D. Birkhoff* for the cases $n = 1, 2, 3, 4, 5$. The proof was made by computations based on tables computed by Cotes. The method is not applicable to the general case.

THEOREM I. *A function $u(x)$ is continuous with its first $2n+2$ derivatives in an interval (a, b), and its $(2n+2)$th derivative itself possesses a derivative for every value of x between a and b. If the function vanishes at the points a and b, and if its first derivative vanishes at $2n+1$ points of the interval, equally spaced and including the end points a and b, then the $(2n+3)$th derivative of the function vanishes between a and b.*

By a successive application of Rolle's theorem it may be inferred directly that $u^{(2n+1)}(x)$ must vanish in the interval (a, b), and that without use of the hypothesis that the points in question are equally spaced. It is important to note, however, that if the points are not equally spaced the conclusion of the theorem is not valid. This fact is made clear by a simple example:

$$u(x) = (x^2-4)^2(4x-1).$$

Here

$$u(2) = u(-2) = u'(2) = u'(-2) = u'(1) = 0.$$

Yet the fifth derivative of $u(x)$ is a constant not zero.

* G. D. Birkhoff, *General mean-value and remainder theorems*, TRANSACTIONS OF THIS SOCIETY, vol. 7 (1906), p. 131.

Suppose that the $2n+1$ equally spaced points have coordinates $x_{-n}, x_{-n+1}, \cdots, x_0, x_1, \cdots, x_n$, and suppose the points arranged from left to right on the line in order of increasing indices.

Then

$$x_{-n} = a, \quad x_n = b.$$

We first make a linear transformation of the independent variable transforming the points in question into the points $-n, -n+1, \cdots, -1, 0, 1, \cdots, n-1, n$. Clearly, if the theorem is valid for this set of points, it is valid for the original set. We may now base our proof on a lemma proved by J. F. Steffensen.*

LEMMA. *If m and n are any positive integers, then the function*

$$(1) \qquad Q(x) = \int_{-m}^{x} x(x^2-1)(x^2-2^2)\cdots(x^2-n^2)\,dx$$

does not vanish in the interval $-m < x < m$.

The proof may readily be supplied by geometrical considerations. We may state our hypotheses on $u(x)$ as follows:

$$u(-n) = u(n) = u'(k) = 0,$$
$$k = -n, -n+1, \cdots, -1, 0, 1, \cdots, n-1, n.$$

Since $u(-n) = 0$,

$$u(x) = \int_{-n}^{x} u'(x)\,dx.$$

Now $u'(x)$ vanishes at the same points as the function

$$P(x) = x(x^2-1)(x^2-2^2)\cdots(x^2-n^2).$$

Determine a function $R(x)$ by the equation

$$u'(x) = P(x)R(x),$$

* J. F. Steffensen, CONFÉRENCES DU CINQUIÈME CONGRÈS DES MATHÉMATICIENS SCANDINAVES, p. 126.

$R(x)$ being defined at an integer r by the equation

$$R(r) = \lim_{x \to r} \frac{u'(x)}{P(x)}.$$

Integrating this equation from $-n$ to n, and recalling that $u(-n) = u(n) = 0$, we have

$$u(n) - u(-n) = \int_{-n}^{n} P(x) R(x)\, dx = 0.$$

On integrating the right hand side of this equation by parts, and setting

$$Q(x) = \int_{-n}^{x} P(x)\, dx,$$

it follows that

$$0 = Q(x) R(x)\Big|_{-n}^{n} - \int_{-n}^{n} R'(x) Q(x)\, dx = \int_{-n}^{n} R'(x) Q(x)\, dx.$$

Now by the lemma, $Q(x)$ is a function of one sign in the interval $-n \leqq x \leqq n$, so that we may apply the first mean-value theorem for integrals. We have thus

$$R'(\zeta) \int_{-n}^{n} Q(x)\, dx = 0, \quad -n < \zeta < n.$$

Now

$$\int_{-n}^{n} Q(x)\, dx \neq 0,$$

since $Q(x)$ does not change sign in the interval $(-n, n)$ and is not identically zero. Hence it follows that

$$R'(\zeta) = 0. \tag{2}$$

We may now make use of Rolle's theorem and of equation (2) to obtain the result desired through the following device. Form the function

$$\Phi(x) = P(x)[R(x) - R(\zeta)].$$

On account of the relation (2), it is seen that the quantity in brackets vanishes at least twice in the point $x = \zeta$. $P(x)$ vanishes $2n+1$ times in the interval $-n \leqq x \leqq n$. Hence,

allowing for multiple zeros, we see that $\Phi(x)$ vanishes at least $2n+3$ times in that interval, whether or not ζ is an integer (that is, a point at which $P(x)$ vanishes). Now by successive applications of Rolle's theorem* we see that $\Phi^{(2n+2)}(x)$ vanishes at least once in the interval $(-n, n)$.

Since the $(2n+2)$th derivative of $P(x)$ is identically zero, and since $P(x)\cdot R(x)$ is $u'(x)$, it follows that

$$\Phi^{(2n+2)}(\xi) = u^{(2n+3)}(\xi) = 0, \quad -n<\xi<n.$$

The theorem is thus established.

3. *A Companion Theorem.* The second theorem is similar in character to the first, but deals with an even number of points where the first theorem dealt with an odd number. The conclusion, however, is essentially different in that the order of the highest derivative whose vanishing can be inferred is one less than the number of conditions imposed on $u(x)$, while in Theorem I these numbers were equal.†

THEOREM II. *A function $u(x)$ is continuous with its first $2n$ derivatives in an interval (a, b), and its $2n$th derivative itself possesses a derivative for every value of x between a and b. If the function vanishes at a and b, and if its first derivative vanishes at $2n$ points of the interval, equally spaced and including the end points a and b, then the $(2n+1)$th derivative of the function vanishes between a and b.*

No loss of generality will be incurred by taking the points as $-2n+1, -2n+3, \cdots, -3, -1, 1, 3, \cdots, 2n-3, 2n-1$. Then $a = -2n+1$, $b = 2n-1$. Since $u(a) = u(b) = 0$, we have

$$\int_a^b u'(x)\,dx = u(b)-u(a) = 0.$$

* See, for example, de la Vallée Poussin, *Cours d'Analyse*, vol. 1, p. 66 (5th edition).

† From the point of view of the general theory of mean-values, as set forth by Birkhoff, this is an essential distinction, Theorem II coming under the general case and Theorem I being an exceptional case.

Set

$$P(x) = (x^2-1)(x^2-3^2)\cdots(x^2-\overline{2n-1}^2),$$

and determine $R(x)$ by the equation

$$u'(x) = P(x)R(x).$$

Then

$$\int_a^b P(x)R(x)dx = 0. \tag{3}$$

Now set

$$q(x) = \frac{P(x)}{x-2n+1}, \qquad Q(x) = \int_a^x q(x)dx.$$

We can show that $Q(x)$ is a function of one sign in the interval (a, b). By the lemma, $Q(x)$ is a function of one sign in the interval $(a, 2n-3)$. (The lemma is geometrical, and is applicable after a change of unit or a shift of origin.) $Q(x)$ vanishes at a and at $2n-3$* but is positive at intermediate points, $q(x)$ is positive in the interval $(2n-3, b)$, and consequently $Q(x)$ is never negative in the interval (a, b).

From equation (3) we have

$$\int_a^b P(x)\,R(x)dx = \int_a^b (x-2n+1)q(x)\,R(x)\,dx = 0.$$

Integrating by parts, we obtain

$$(x-2n+1)R(x)Q(x)\Big|_b^a - \int_a^b \frac{d}{dx}[(x-2n+1)R(x)]\,Q(x)\,dx = 0.$$

The first term on the left hand side drops, so that we have on applying the first mean-value theorem for integrals

$$\frac{d}{dx}\Big[(x-2n+1)R(x)\Big]_{x=\zeta}\int_a^b Q(x)\,dx = 0, \qquad a<\zeta<b.$$

Since $\int_a^b Q(x)dx \neq 0$, it follows that

$$\frac{d}{dx}\Big[(x-2n+1)R(x)\Big]_{x=\zeta} = 0. \tag{4}$$

* This follows because $q(x-1) = q(-x-1)$.

We now employ a device similar to that used in the proof of Theorem I. Set

$$\Phi(x) = P(x)R(x) - R(\zeta)(\zeta - 2n + 1)q(x)$$
$$= q(x)[R(x)(x - 2n + 1) - R(\zeta)(\zeta - 2n + 1)].$$

Now on account of the relation (4) it is seen that the quantity in brackets has at least two zeros in the point $x = \zeta$; and $q(x)$ has $(2n - 1)$ zeros. It follows then that $\Phi(x)$ has at least $(2n + 1)$ zeros in the closed interval (a, b). By Rolle's theorem $\Phi^{(2n)}(x)$ must vanish at least once in that interval. Since $P(x)R(x)$ is $u'(x)$, and since the $2n$th derivative of $q(x)$ is identically zero, it follows that

$$\Phi^{(2n)}(\xi) = u^{(2n+1)}(\xi) = 0, \qquad a < \xi < b.$$

The theorem is thus established.

4. *Applications to Mechanical Quadrature.* Let us now apply these theorems to the problem of mechanical quadrature. Suppose a function $f(x)$ is known at equally spaced points $x_1, x_2, x_3, \cdots x_{2n+1}$, and suppose it be required to find an approximate expression for

$$\int_{x_1}^{x_{2n+1}} f(x)\,dx.$$

Let $F(x)$ be the polynomial of degree $2n$ at most, taking on the known values of $f(x)$ at the given points. Then form the function

$$u(x) = \int_{x_1}^{x} [f(x) - F(x)]\,dx - A\int_{x_1}^{x} P(x)(x - x_{n+1})\,dx,$$

where

$$P(x) = (x - x_1)(x - x_2) \cdots (x - x_{2n+1}),$$

and where A is to be so determined that $u(x_{2n+1}) = 0$. This is possible since

$$\int_{x_1}^{x_{2n+1}} P(x)(x - x_{n+1})\,dx \neq 0.*$$

* This may be seen by writing

$$\int_{x_1}^{x_{2n+1}} P(x)(x - x_{n+1})\,dx = -\int_{x_1}^{x_{2n+1}} dx \int_{x_1}^{x} P(t)\,dt$$

and then applying the Lemma.

Now $u(x)$ satisfies all the conditions of Theorem I (making proper assumptions regarding the continuity of $f(x)$, and consequently

$$u^{(2n+3)}(\xi) = 0, \qquad x_1 < \xi < x_{2n+1}.$$

That is,

$$f^{(2n+2)}(\xi) - A(2n+2)! = 0, \quad A = f^{(2n+2)}(\xi)/(2n+2)!$$

Since $u(x_{2n+1}) = 0$, we have

$$\int_{x_1}^{x_{2n+1}} f(x)\,dx = \int_{x_1}^{x_{2n+1}} F(x)\,dx + \frac{f^{(2n+2)}(\xi)}{(2n+2)!}\int_{x_1}^{x_{2n+1}} P(x)(x-x_{n+1})\,dx, \tag{5}$$

a remainder formula for mechanical quadrature.

In case the number of points is even, we use theorem II and obtain

$$\int_{x_1}^{x_{2n}} f(x)\,dx = \int_{x_1}^{x_{2n}} F(x)\,dx + \frac{f^{(2n)}(\xi)}{(2n)!}\int_{x_1}^{x_{2n}} (x-x_1)\cdots(x-x_{2n})\,dx. \tag{6}$$

These formulas were given by Birkhoff in the paper already cited, but were established only for small values of n. Steffensen established formula (5) by other methods in the article above cited.

5. *Extensions.* The methods here employed might be used to prove certain other mean-value theorems involving equally spaced points, and to which Rolle's theorem is not directly applicable. For example, I have proved the following theorem.

THEOREM III. *A function $u(x)$ having the same amount of continuity as that of Theorem* I *vanishes at the points a and b. If (a, b) is divided into $2n+2$ equal parts, and if $u'(x)$ vanishes at the interior points of division, then $u^{(2n+3)}(x)$ vanishes at an interior point of (a, b).*

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THE GEOMETRY OF FREQUENCY FUNCTIONS*

BY DUNHAM JACKSON

1. *Introduction.* The Pearson coefficient of correlation, calculated for a finite number of observations, has a geometric interpretation which is simple and almost immediate.† The same thing may be said of the corresponding expression formed for a pair of functions of a continuous variable.‡ When the distribution of the observed quantities is thought of as given by a frequency function, the geometric interpretation of the correlation coefficient is not so obvious. It is the purpose of this paper to show one form that such an interpretation may take.§ The geometric configurations are exactly the same as in the other cases mentioned; the difference is in the manner of setting up the association. This is accomplished by defining an appropriate correspondence between an arbitrary point of a plane, or of space, and an arbitrary linear combination of the variables subjected to measurement.

There will be no assumption that the distributions involved are "normal", in the sense of the Gaussian law. There will be incidental reference to frequency functions having properties that correspond to those of normal orthogonal sets of functions, as the terms are used in the theory of the development of arbitrary functions in series; but

* Presented to the Society, October 25, 1924.

† Cf., e. g., D. Jackson, *The trigonometry of correlation*, AMERICAN MATHEMATICAL MONTHLY, vol. 31 (1924), pp. 275–280; also the paper cited in the next footnote.

‡ Cf., e. g., D. Jackson, *The elementary geometry of function space*; recently submitted to the AMERICAN MATHEMATICAL MONTHLY.

§ For another form, cf. James McMahon, *Hyperspherical goniometry; and its application to correlation theory for n variables*, BIOMETRIKA, vol. 15 (1923), pp. 173–208. The fundamental idea of attaching a geometric meaning to the correlation coefficient appears to be due to Pearson himself.

the word "normal" will not be used in this connection, because of the possibility of misunderstanding. As the paper is concerned primarily with the establishment of certain formal relations, questions of convergence will be avoided for the most part by assuming that the functions considered are different from zero only over a finite range; the concluding paragraph will deal briefly with distributions that "tail off" to zero at infinity.

The order of presentation in the main part of the paper is closely parallel to that followed in the author's article on *The elementary geometry of function space,* to which reference has been made in a previous footnote. There is enough difference in detail, however, to call for an independent treatment at some length.

2. *Frequency Functions in Two Variables.* To begin with the case of two variables, let $\varphi(x, y)$ be defined and continuous as a function of its arguments over a finite region R of the plane; let it be everywhere positive or zero, and not identically zero. There is no essential loss of generality, and there is some gain of simplicity, in assuming that

$$\int\int_R \varphi(x, y)\,dx\,dy = 1.$$

For preliminary consideration, let it be supposed further that φ satisfies the conditions*

$$(1) \qquad \int xy\varphi = 0, \qquad \int x^2\varphi = \int y^2\varphi = 1,$$

where $\int xy\varphi$ is an abbreviated notation for the double integral

$$\int\int_R xy\varphi(x, y)\,dx\,dy,$$

and where the other integrals are to be interpreted similarly.

Let a and b be any two real numbers. Then we have $\int(ax+by)^2\varphi = a^2+b^2$, because of (1); if (a, b) are taken

* A method of constructing an infinite variety of functions φ satisfying the conditions imposed will appear presently.

as the coordinates of a point, the value of the integral is the square of the distance of the point from the origin. If P_1 and P_2 are two points with coordinates (a_1, b_1) and (a_2, b_2), and O the origin, the distance P_1P_2 is the square root of the quantity

$$\int [(a_2-a_1)x+(b_2-b_1)y]^2\varphi = (a_2-a_1)^2+(b_2-b_1)^2,$$

and the cosine of the angle P_1OP_2 is

$$\frac{a_1a_2+b_1b_2}{\sqrt{a_1^2+b_1^2}\,\sqrt{a_2^2+b_2^2}} = \frac{\int (a_1x+b_1y)(a_2x+b_2y)\varphi}{\sqrt{\int (a_1x+b_1y)^2\varphi}\,\sqrt{\int (a_2x+b_2y)^2\varphi}}$$

Thus the fundamental geometric measures can be expressed, somewhat indirectly and artificially to be sure, in terms of integrals involving the function φ.

In transition, let φ be a function subject to the same hypotheses as before, except that the last two of the conditions (1) are not imposed; and let*

$$\sqrt{\int x^2\varphi} = \sigma, \qquad \sqrt{\int y^2\varphi} = \tau.$$

If new variables ξ, η are introduced by means of the relations $\xi = x/\sigma$, $\eta = y/\tau$, so that, incidentally, $d\xi = dx/\sigma$, $d\eta = dy/\tau$, and if the function $\sigma\tau\varphi(x, y)$ is designated, with regard to its dependence on the new variables, by $\Phi(\xi, \eta)$, it is found that

$$(2)\quad \begin{cases} \int\int \Phi(\xi,\eta)\,d\xi\,d\eta = 1, & \int\int \xi\eta\,\Phi(\xi,\eta)\,d\xi\,d\eta = 0, \\ \int\int \xi^2\Phi(\xi,\eta)\,d\xi\,d\eta = \int\int \eta^2\Phi(\xi,\eta)\,d\xi\,d\eta = 1, \end{cases}$$

* Since $\int\varphi = 1$, the quantities σ and τ are the standard deviations of x and y, when φ is interpreted as a frequency function, if $\int x\varphi = \int y\varphi = 0$; these last conditions have no bearing on the formal work in hand, apart from its statistical interpretation, but they may be thought of as included among the conditions imposed on φ, if it appears that the notation is likely to cause confusion otherwise.

the integrals being extended over the region of definition of Φ. A simple substitution has replaced φ by a function satisfying all the original hypotheses.

Now let φ, while retaining the other properties that have been assigned to it, be relieved of the restrictions (1) altogether. Let

$$\sqrt{\int x^2\varphi} = \sigma, \quad \int xy\varphi = p, \quad y' = y - (p/\sigma^2)x, \quad \sqrt{\int y'^2\varphi} = \tau'.$$

It is found by substitution that

$$\int xy'\varphi = \int xy\varphi - \frac{p}{\sigma^2}\int x^2\varphi = 0.$$

As a matter of notation, let $x' = x$, $\sigma' = \sigma$, $\varphi(x,y) = \overline{\varphi}(x',y')$. Since the functional determinant of x' and y' with respect to x and y is 1, three of the above relations may be written in the form

$$\int\int x'^2\overline{\varphi}(x',y')\,dx'\,dy' = \sigma'^2, \quad \int\int y'^2\overline{\varphi}(x',y')\,dx'\,dy' = \tau'^2,$$

$$\int\int x'y'\overline{\varphi}(x',y')\,dx'\,dy' = 0,$$

the range of integration now being that over which $\overline{\varphi}(x',y')$ is defined. The present $\overline{\varphi}$ then corresponds to the function φ in the preceding paragraph. To obtain a Φ satisfying (2), it is sufficient to let

$$\xi = x'/\sigma', \quad \eta = y'/\tau', \quad \sigma'\tau'\overline{\varphi}(x',y') = \Phi(\xi,\eta).$$

The original variables x, y are expressed in terms of ξ, η by the equations

$$x = \sigma'\xi, \quad y = \frac{p}{\sigma'}\xi + \tau'\eta, \tag{3}$$

which are of the general form

$$x = a_1\xi + b_1\eta, \quad y = a_2\xi + b_2\eta. \tag{4}$$

The particular determination of the coefficients specified in (3) is only one of an infinite variety of determinations which will serve essentially the same purpose. For if ξ

and η are subjected to a further transformation defined by setting

$$\xi' = \alpha_1\xi + \beta_1\eta, \quad \eta' = \alpha_2\xi + \beta_2\eta,$$

where

$$\alpha_1^2 + \beta_1^2 = \alpha_2^2 + \beta_2^2 = 1, \quad \alpha_1\alpha_2 + \beta_1\beta_2 = 0,$$

and if $\Phi(\xi, \eta) = \bar{\Phi}(\xi' \eta')$, equations of the form (2) are satisfied in terms of the new variables ξ' and η', because of the equations (2) themselves, in terms of ξ and η, and the fact that the functional determinant* of the transformation is ± 1. There are accordingly infinitely many transformations of the form (4) which yield a Φ satisfying (2), if $\Phi(\xi, \eta)$ is equal to $\varphi(x, y)$ multiplied by the absolute value of the determinant of the coefficients. Any transformation (4) fulfilling this requirement may be taken as a basis for the work that is to follow, and it will be understood that the symbols a_1, b_1, a_2, b_2 refer to the coefficients in such a transformation.

With this understanding, it is seen at once that

$$\int x^2\varphi = \int\int (a_1\xi + b_1\eta)^2 \Phi(\xi, \eta)\, d\xi d\eta = a_1^2 + b_1^2,$$

$$\int y^2\varphi = a_2^2 + b_2^2,$$

$$\int xy\varphi = \int\int (a_1\xi + b_1\eta)(a_2\xi + b_2\eta)\, \Phi(\xi, \eta)\, d\xi d\eta = a_1a_2 + b_1b_2;$$

when the variables of integration are not expressly indicated, it is intended always that the integration shall be performed with regard to x and y. *The equations* (4) *serve, when the function* φ *is given, to associate* x *and* y *with two points* P_1 *and* P_2 *in the* (ξ, η)*-plane, having the coordinates* (a_1, b_1) *and* (a_2, b_2) *respectively. Because of the degree of arbitrariness retained by the coefficients in* (4), *the association may*

* To repeat the familiar proof,

$$\begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix}^2 = \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} \begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix} = \begin{vmatrix} \alpha_1^2 + \beta_1^2 & \alpha_1\alpha_2 + \beta_1\beta_2 \\ \alpha_1\alpha_2 + \beta_1\beta_2 & \alpha_2^2 + \beta_2^2 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

be carried out in an infinite variety of ways, but in all cases the expressions

$$(5) \qquad \sqrt{\int x^2\varphi}, \quad \sqrt{\int y^2\varphi}, \quad \frac{\int xy\varphi}{\sqrt{\left(\int x^2\varphi\right)\left(\int y^2\varphi\right)}},$$

which are themselves independent of the coefficients, represent respectively the distance OP_1, the distance OP_2, and the cosine of the angle P_1OP_2. The last expression, in case $\int x\varphi = \int y\varphi = 0$, is the coefficient of correlation between two variables x and y distributed according to the frequency function φ, and a geometric meaning has thus been assigned to the correlation coefficient.

The correspondence that has been established renders it possible to prove analytical theorems involving a frequency function by the mere interpretation of geometric facts. Consider, for example, the problem of determining a coefficient λ so as to make the value of the integral $\int(y-\lambda x)^2\varphi$ a minimum. Since this integral is equal to

$$\int\int [(a_2-\lambda a_1)\xi + (b_2-\lambda b_1)\eta]^2\, \Phi(\xi,\eta)\, d\xi d\eta,$$

it is the square of the distance between the points $(\lambda a_1, \lambda b_1)$ and (a_2, b_2). The points (a_1, b_1) and (a_2, b_2) have already been denoted by P_1 and P_2; let Q stand for the point $(\lambda a_1, \lambda b_1)$, and θ for the angle P_1OP_2. Then Q is on the line OP_1, the distances OQ and OP_1 being in the ratio of λ to 1. In order that the distance QP_2 may be a minimum, Q must be the foot of the perpendicular from P_2 on OP_1. This means that $\overline{OQ} = \overline{OP_2}\cos\theta$, and

$$\lambda = \frac{\overline{OQ}}{\overline{OP_1}} = \frac{\overline{OP_2}}{\overline{OP_1}}\cos\theta$$

$$= \frac{\sqrt{\int y^2\varphi}}{\sqrt{\int x^2\varphi}} \cdot \frac{\int xy\varphi}{\sqrt{\left(\int x^2\varphi\right)\left(\int y^2\varphi\right)}} = \frac{\int xy\varphi}{\int x^2\varphi}.$$

If $\int x\varphi = \int y\varphi = 0$, the value of $\int \varphi$ being 1 throughout, the quantities

$$\sigma = \sqrt{\int x^2\varphi}, \qquad \tau = \sqrt{\int y^2\varphi}$$

are the *standard deviations* of x and y respectively, while, as already mentioned, $\cos\theta = r$ is the *coefficient of correlation* of x and y. With this notation, it is seen that $\lambda = (\tau/\sigma)r$. The minimum distance QP_2 is equal to

$$\overline{OP_2}\sin\theta = \tau\sqrt{1-r^2}.$$

The familiar determination of the coefficient of regression of y on x and of the root-mean-square deviation from the line of regression will be recognized at once.

3. *Frequency Functions in Three Variables.* The geometric treatment can be extended to three dimensions without difficulty. To begin, as before, with a preliminary inspection of a special case, let $\varphi(x, y, z)$ be defined throughout a finite region of space, continuous, everywhere positive or zero, not identically zero, and so constituted that

$$(6) \qquad \int xy\varphi = \int xz\varphi = \int yz\varphi = 0,$$
$$\int \varphi = \int x^2\varphi = \int y^2\varphi = \int z^2\varphi = 1,$$

the integrals being triple integrals with regard to x, y, and z, extended over the region of definition of φ. If P_1 and P_2 are two points in space, with coordinates (a_1, b_1, c_1) and (a_2, b_2, c_2), O being the origin, it is found directly that the squares of the distances OP_1, OP_2, P_1P_2 are represented by the integrals

$$\int (a_1x + b_1y + c_1z)^2\varphi, \qquad \int (a_2x + b_2y + c_2z)^2\varphi,$$
$$\int [(a_2 - a_1)x + (b_2 - b_1)y + (c_2 - c_1)z]^2\varphi.$$

A corresponding evaluation of the quotient

$$\frac{\int (a_1x+b_1y+c_1z)\,(a_2x+b_2y+c_2z)\varphi}{\sqrt{\int (a_1x+b_1y+c_1z)^2\varphi}\ \sqrt{\int (a_2x+b_2y+c_2z)^2\varphi}}$$

gives the familiar formula of solid analytic geometry for the cosine of the angle P_1OP_2.

If the last three equations in (6) are replaced by

$$\sqrt{\int x^2\varphi} = \sigma, \qquad \sqrt{\int y^2\varphi} = \tau, \qquad \sqrt{\int z^2\varphi} = \omega,$$

the unit values of the integrals may be restored by setting

$$\xi = x/\sigma, \quad \eta = y/\tau, \quad \zeta = z/\omega, \quad \sigma\tau\omega\varphi(x, y, z) = \Phi(\xi, \eta, \zeta).$$

If φ is not subject to (6) at all, except for the condition $\int\varphi = 1$, a reduction may be performed as follows. Let

$$x' = x, \qquad \sqrt{\int x'^2\varphi} = \sigma', \qquad \int x'y\varphi = p'_{12},$$

$$y' = y - (p'_{12}/\sigma'^2)x', \quad \sqrt{\int y'^2\varphi} = \tau', \quad \int x'z\varphi = p'_{13}, \quad \int y'z\varphi = p'_{23},$$

$$z' = z - (p'_{13}/\sigma'^2)x' - (p'_{23}/\tau'^2)y', \quad \sqrt{\int z'^2\varphi} = \omega',$$

$$\varphi(x, y, z) = \overline{\varphi}(x', y', z').$$

It is readily verified that

$$\int\int\int x'y'\overline{\varphi}(x', y', z')dx'dy'dz'$$

$$= \int\int\int x'z'\overline{\varphi}(x', y', z')dx'dy'dz'$$

$$= \int\int\int y'z'\overline{\varphi}(x', y', z')dx'dy'dz' = 0.$$

It remains to set

$$\xi = x'/\sigma', \qquad \eta = y'/\tau', \qquad \zeta = z'/\omega',$$

$$\sigma'\tau'\omega'\overline{\varphi}(x', y', z') = \Phi(\xi, \eta, \zeta),$$

and the desired reduction is accomplished; that is, Φ has the properties that

$$\begin{aligned}
&\iiint \Phi d\xi d\eta d\zeta = 1,\\
&\iiint \xi\eta\Phi d\xi d\eta d\zeta = \iiint \xi\zeta\Phi d\xi d\eta d\zeta\\
&\qquad = \iiint \eta\zeta\Phi d\xi d\eta d\zeta = 0, \qquad (7)\\
&\iiint \xi^2\Phi d\xi d\eta d\zeta = \iiint \eta^2\Phi d\xi d\eta d\zeta\\
&\qquad = \iiint \zeta^2\Phi d\xi d\eta d\zeta = 1.
\end{aligned}$$

Let it be understood henceforth that x, y, z are related to a new set of variables ξ, η, ζ, by equations of the form

$$\begin{aligned}
x &= a_1\xi + b_1\eta + c_1\zeta,\\
y &= a_2\xi + b_2\eta + c_2\zeta,\\
z &= a_3\xi + b_3\eta + c_3\zeta,
\end{aligned}$$

so that the conditions (7) are satisfied, when $\Phi(\xi, \eta, \zeta)$ is equal to $\varphi(x, y, z)$ multiplied by the absolute value of the determinant of the coefficients. The coefficients may be determined by the calculation of the preceding paragraph, or in any one of an infinite variety of other ways, the passage from one determination to another amounting to a rotation of coordinate axes, with or without a reflection in one of the coordinate planes. Let the point with coordinates (a_i, b_i, c_i) be denoted by P_i, for $i = 1, 2, 3$, and the origin, as usual, by O. The expressions (5) once more represent the distances OP_1 and OP_2 and the cosine of the angle P_1OP_2, while the corresponding expressions obtained by permuting the variables are to be similarly interpreted.

A foundation is thus laid for geometric reasoning on a more extensive scale. If U and V are any two linear combinations of x, y, and z,

$$U = l_1x + m_1y + n_1z, \qquad V = l_2x + m_2y + n_2z,$$

they can be expressed in the form

$$U = A_1\xi + B_1\eta + C_1\zeta, \qquad V = A_2\xi + B_2\eta + C_2\zeta,$$

and then $\int U^2\varphi$ is the square of the distance of the point (A_1, B_1, C_1) from the origin, $(\int UV\varphi)/\sqrt{(\int U^2\varphi)(\int V^2\varphi)}$ is the cosine of the angle subtended at the origin by the points (A_1, B_1, C_1) and $A_2, B_2, C_2)$, and so on. Any linear combination of x, y, and z corresponds to a definite point in three-dimensional space.

Suppose λ and μ are determined so as to minimize the integrals $\int(y-\lambda x)^2\varphi$ and $\int(z-\mu x)^2\varphi$. The points Q_2 and Q_3 corresponding to λx and μx are on the line OP_1, where this line is met by the perpendiculars from P_2 and P_3 respectively. Let $u = y-\lambda x$, $v = z-\mu x$; the coordinates of the points corresponding to u and v are the components of the vectors $\underline{Q_2P_2}$ and $\underline{Q_3P_3}$. Consequently the expression $(\int uv\varphi)/\sqrt{(\int u^2\varphi)(\int v^2\varphi)}$ is equal to the cosine of the dihedral angle between the planes P_1OP_2 and P_1OP_3. If $\int x\varphi = \int y\varphi = \int z\varphi = 0$, it is at the same time the coefficient of partial correlation between y and z. The geometric figure is exactly the same as in the case of a finite number of observations, and there is no need of repeating the steps by which the coefficients of partial correlation and of double correlation are calculated in terms of ordinary correlation coefficients.* It may be verified that the determinants

$$\begin{vmatrix} \int x^2\varphi & \int xy\varphi \\ \int yx\varphi & \int y^2\varphi \end{vmatrix}, \qquad \begin{vmatrix} \int x^2\varphi & \int xy\varphi & \int xz\varphi \\ \int yx\varphi & \int y^2\varphi & \int yz\varphi \\ \int zx\varphi & \int zy\varphi & \int z^2\varphi \end{vmatrix}$$

represent respectively the square of the area of the

* Cf. the papers on *The trigonometry of correlation* and *The elementary geometry of function space*, previously cited.

parallelogram determined by OP_1 and OP_2, and the square of the volume of the parallelepiped determined by OP_1, OP_2, and OP_3. The values of the determinants, which have the character of Gramian determinants, are therefore always positive.*

4. *Functions Defined over an Infinite Range.* As was said in the introduction, it is not the purpose of this paper to enter into an elaborate discussion of the questions of convergence that arise if the frequency functions considered are supposed defined and different from zero over an infinite range, in a highly arbitrary manner. There is no difficulty, however, in arriving at a formulation general enough to cover the most important statistical applications. Suppose, for example (to concentrate attention on the case of three variables), that

$$\varphi(x,y,z) < Ke^{-|x|-|y|-|z|}$$

for all values of x, y, and z, K being a constant. Then all the integrals that appear in the course of the discussion are absolutely convergent, so that the integral over a finite region approaches a definite limit as the boundary of the region recedes to infinity in any way whatever. To justify the transformations of variable in the infinite integrals, it is sufficient to carry out the transformations over corresponding finite regions, a sphere, for example, in one set of variables, and an ellipsoid in the other set, and then to allow these regions to expand to infinity. Furthermore, though this is not essential to the argument, all the triple integrals can be evaluated as iterated integrals.

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* The fact that the two-rowed determinant is positive is equivalent to the fact that the value of a coefficient of correlation is always between -1 and $+1$. It is to be noticed that the extreme values $+1$ and -1 can not occur, under the hypotheses on which this paper has been based; a frequency function corresponding to perfect positive or negative correlation is not continuous as a function of the two or more variables involved.

MATHEMATICS FOR ENGINEERS

Lezioni di Analisi Matematica. By Guido Fubini. Torino, Società Tipografico-Editrice Nazionale, 1923. viii + 470 pp.

Grundriss der Differential Rechnung. By Ludwig Kiepert. Hannover, Helwingsche Verlagsbuchhandlung, 1923. viii + 348 pp. II. Band: *Einige grundlegende Untersuchungen aus der Algebra und Funktionen von mehreren unabhängigen Veränderlichen.*

Both of these books were written primarily for engineering and technical students. The former follows the course as given in the Polytechnic school at Turin, and the latter is written by a trained engineer for engineering students. It has been remarked so often that our scientific students do not know enough mathematics to enable them to go very far; particularly is this true of our physicists and engineers. A Ph. D. in chemistry complains that he cannot read beyond the first hundred pages of Sommerfeld's *Structure of the Atom and Spectral Lines*, because there is too much mathematics. How far this is true may be judged by comparing the mathematical equipment of the above courses with one obtained in a calculus course in an American university.

The present edition of the first book is the fourth; previous editions appeared in 1913, 1915, 1920, which indicates its wide use in Italy. The difference in content between these texts and our American texts is not so marked. There is perhaps a greater emphasis on applications of calculus to algebraic problems in these foreign books, but the chief difference is in the point of view.

It is rather strange that we should be laboring under severe handicaps in the teaching of the pivotal course of our mathematical curriculum, the calculus. Our difficulties are two-fold. First, there is in vogue, what I may call the "double standard" amongst us. Mathematicians of repute will not hesitate to pursue a policy in their teaching which amounts to keeping their right hands in ignorance of what their left hands are doing. Two examples will illustrate this difficulty. An author of a calculus text writes in a footnote that the equation in the paragraph above "to be strictly accurate" requires some modification. Again, how often do we find theorems stated as true, which in the proof make use of a uniformity condition, which bobs up serenely from nowhere, and which disappears again in applications to nowhere.

Our second difficulty is that too many of us bury our heads ostrich-like in the sand and keep the eyes of our students closed to the fundamental assumptions on which the basic principles of the course are founded. This tends to superficiality and the "hand-book" method

of teaching the calculus. As a student, the reviewer recalls emerging from the first calculus course with no other knowledge of what it was all about than being able to differentiate x^3 and integrate x^2dx. Our first course does not go deeply enough into the facts.

If the "small zeros" of the engineer are ever to be banished from the face of the earth, it will be due to the fact that our teaching will from the very beginning have taught him the significance of relative size. No opportunity will be given him to form bad mathematical habits and to acquire erroneous ideas. No longer will we write equations whose only interpretation is "that one variable is approaching another *variable* as a limit".

It is quite a contrast to regard the first book, especially, that by Fubini, and to see what its bearing is on our situation. The author aims at the outset to set forth the fundamental notions of number, angle, area and volume. There are no extensive proofs given, but the facts are stated correctly and interestingly. There follows about 100 pages of algebra, principally the theory of equations, which an American student would normally acquire in his freshman or sophomore year. The remainder of the book deals with the usual topics of the calculus. The meaning of the principles is profusely illustrated by means of examples, taken chiefly from the field of mechanics on account of the future interest of the student. The formal work of differentiation is not given very much space, but all the essential formulas are developed. The applications of the derivative are the usual ones to curve sketching, indeterminate forms, multiple roots of an algebraic equation, maxima and minima and points of inflexion. The first and second laws of the mean for derivatives are proved.

The definite integrals, simple and multiple, are introduced by means of additive functions of a one-dimensional, two-dimensional, or three-dimensional region, as the case may be. The proof for integrability is based on an argument which is very similar to the one given by the reviewer,* except that Fubini's theorem† is not as general. His final result stated on page 329 requires uniformity and is equivalent to Osgood's‡ form of Duhamel's theorem.

The latter half of the course includes what would be included in

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* H. J. Ettlinger, *A simple form of Duhamel's theorem and some new applications*, AMERICAN MATHEMATICAL MONTHLY, vol. 29 (1922), p. 246 ff. See also a paper by the reviewer, *On the integrability of a continuous function*, AMERICAN MATHEMATICAL MONTHLY, vol. 31 (1924), p. 319.

† Cf. pp. 316–17 and footnote of the book under review.

‡ W. F. Osgood, *The integral as a limit of a sum and a theorem of Duhamel's*, ANNALS OF MATHEMATICS, (2), vol. 4 (1903), pp. 164–65.

a course of advanced calculus in this country. The existence theorem due to Cauchy for differential equations is stated, but its proof is omitted. Space geometry, envelopes, line and surface integrals, transformation of a double integral, Fourier series, and functions of a complex variable are touched upon and treated to a slight extent in this part of the book.

There is one list of problems thirty-nine in number on pp. 55–61 for the chapter on theory of equations. The exercises in the rest of the book are few and far between. It is probably true that long lists of calculus problems are available for class use from other independent sources.

The printing of the book is unusually good; only two misprints have been noticed. On page 123, near the bottom of the page, the third term of the geometric progression should be $1/2^2$ instead of $1/3$. On page 170, the term $1/x$ is omitted from the formula for the derivative of $\log_e x$.

The book by Kiepert deals with the applications of the calculus to algebra, particularly the solution of equations numerically to any required degree of approximation. In the latter half of the book, functions of several variables are treated with applications to space analytic geometry, including what is ordinarily included in differential geometry in this country.

Throughout the book the complete solution of a great many examples is given to illustrate the principles used. The last chapter is devoted to maxima and minima of functions of several variables. There is a summary of the important formulas proved in this volume at the end of the book.

The plan of both of these books commends itself very highly to those who are desirous of improving the teaching of the calculus not alone for those who use mathematics as a tool, but also for those whom we may look to as the future searchers after mathematical truth.

H. J. ETTLINGER

STUDY ON VECTORS AND INVARIANTS

Einleitung in die Theorie der Invarianten linearer Transformationen auf Grund der Vektorenrechnung. Erster Teil. By E. Study. Braunschweig, Vieweg und Sohn, 1923. 268 pp.

Like other writings of Professor Study the present work is interesting from several different angles. It is a development of the theory of invariants of ternary forms, based upon what he calls vectors; it is an exposition of his thesis that mathematics is the study* of "natural (positive, integral) numbers and everything which can be based upon them, but nothing else"; it is a running commentary and criticism of many things and many mathematicians. One needs occasionally a background of other papers of his dealing with questions of a semi-philosophical character. He particularly takes to task the investigators who have developed systems of vector analysis because they have not contented themselves with utilising the theory of invariants, as developed by Clebsch, Aronhold, etc. He says (p. 3) "Whether one develops vector analysis for the sake of its applications or as a self-contained discipline, there must always come first, as stated, a consideration of certain topics of algebra, namely invariants of certain groups of linear transformations: all expressions considered are invariants of orthogonal transformations, or invariants of groups closely related to the group of orthogonal transformations. But for more than fifty years we have had a highly developed theory of the group of all linear transformations, and for more than twenty-five years at least the fundamentals of a theory of invariants of the other groups just referred to (Ges. Wiss. Leipzig, Math.-Phys. 1897, p. 443 ff.). But not a glimmer of light seems to have fallen from these investigations upon the highly beloved "Vector Analysis" of today. Indeed old problems have been handled as if never treated before, and we are thus lagging far behind what has long been a guaranteed possession of the science. So far as I know, the question is never raised in such writings, as to what are all possible algebraic, and in particular, rational, invariants of these groups; and yet there can be no doubt that this is the problem, which from the very nature of things, lies at the heart of the matter." And again "As for the majority of authors it is not evident that they lived in a generation when the theory of groups was in full bloom. We even see the algorithm of H. Grassmann, which was a mark of progress in his own time, and which stands as a monument to his originality, yet which has long ago been absorbed into the more profound theory of invariants, hailed as the acme of the attainable, or even as a panacea, which certainly it is not; while on the

* *Mathematik and Physik*, p. 6.

other hand algebraists like Aronhold and Clebsch, who, differing from their predecessors, have set out from precisely stated problems, and have gone much farther, do not seem to have existed for these same authors. In short we are behind the times, and nothing less."

The Introduction, of twelve full pages, is labeled "Problems and methods", and is an insistence upon the importance of problems and the relative unimportance of methods. "Those who construct vector analysis and related algorithms whose essence is method (extensive algebra, calculus of matrices, quaternions, etc.) usually consider their subject-matter as a world in itself. Doubtless there is much justification for this. Such a claim and the related desire for purity of method contain a wholesome pressure towards thoroughness". There are esthetic considerations but there is also the usefulness and ease of manipulation of an algorithm to be taken into account. A happy notation is a great advantage in the progress of a new branch of mathematics, but for that very reason one must hesitate to introduce novelties merely for their own sake, or from a mere desire not to "violate the spirit of the order". The reader will not usually learn a new notation thoroughly, and will in the end give up trying to translate it into what he is familiar with. This also has to be said with regard to the invention of new terms, as, for instance, calling ternary bilinear forms, or the system of coefficients that belong to such a form, by the new terms: tensor of the second kind, affinor, dyadic, tensortriple, complete dyad, asymmetric tensor, diatensor, vector homography, and special terms related, as deviator, antitensor, axiator, idemfactor, versor, perversor, etc. And "what has long been sought in vector calculus—a system of symbols most convenient for the subject,—has been in existence a long time, in a form much more convenient than anything devised since by the various partisans. I shall make use here with minor alterations (whose motive should not be missed) of the notation based upon the older theory of invariants of linear transformations, which is uniquely determined in all essentials by the subject itself. A special notation for vectors, which only too easily may become an obstacle in the free road, seems to me superfluous if not harmful. The 'world for itself' will nevertheless come into its full rights." Professor Study thus comes to the conclusion that the problems of vector calculus are merely invariant problems, and the methods of invariants are the most satisfactory for all purposes of vectors. Thus the raison d'être of vector algorithms has disappeared!

Professor Study also has no uncertain opinion with regard to many ideas of many mathematicians. On page 3, Grassmann is said to have been "absorbed". On page 4 an unnamed writer* is criticised for apologiz-

* See Weyl, MATHEMATISCHE ZEITSCHRIFT, vol. 20 (1924), p. 131.

ing for the deluge of formulas he introduces. On page 5 another unnamed writer is criticised for his remark relative to Hilbert's theorem on the finiteness of systems of forms: "How fine that we need not bother any more with invariants." One must feel that perhaps the point was missed. On page 8, Boltzmann is mentioned as saying: "One should leave elegance to cobblers and tailors". On page 11, Beck is criticised for his *Coordinate Geometry*, though he certainly is in sympathy with Study's ideas. On page 19 and page 21, Lie is mentioned as illogical for his term "continuous group", since Study desires to use the term in another sense. On page 52, Coolidge gets favorable mention, as do Veblen and Young on page 59. On page 101, Grassmann's Lückenprodukte are condemned. On the same page the mathematical ability of Einstein is questioned, and Weyl's notations are condemned. On page 118, the right of authors to invent terms is questioned. On page 127, Frobenius comes in for a share of criticism, along with unnamed mathematicians who use the English language, and spend time on hypernumbers. The criticism of Frobenius' standpoint extends over to page 129. On page 238, authors of analytic geometries are criticised. On page 257, Rabinovitch and the Italian school get their deserts.

Professor Study's definitions remove geometry from the study of space, as usually conceived, completely. For him a point is merely a triplex of numbers, or in general a multiplex of N numbers, and a vector is precisely the same thing. The term multiplex, it may be explained, means a set of numbers in a specified order. It is exactly the idea of Hamilton over again, who first dealt with couples, then with "sets" of numbers. Of course Study would and does repudiate the Hamilton relation of number to "time", but as a matter of fact Hamilton's "time" is nothing more than the linear continuum. He does not point out, however, that it was Hamilton who first made a study of the couple, multiplex set, etc., as an entity. He did not call it a vector, defining (for the first time) this to mean a geometric segment with a definite direction. One who insists on the correct use of established terminology should have retained the very useful terms of Hamilton. The reviewer agrees perfectly with Professor Study that good existing terminologies should be preserved, and new ones avoided; therefore he desires Professor Study to follow the same caution. The same remark applies to changing the terminology of Lie. And it is to be noted further that one does not escape axioms and postulates by merely referring everything to the system of natural numbers. If that were possible he could go still further and start with Whitehead and Russell's *Principia*. We might grant that it is possible to set up a useful isomorphism between points or vectors in space and triplexes, but it is easily evident that the isomorphism will work both ways, and that arithmetic can be based upon space just as much as space can be based

upon arithmetic. A good deal of so-called simplification and "going to the foundations" consists in substituting one abstraction for another.

But we also have a conviction that the theory of vectors is not the same by any means as the theory of invariants. Professor Study seems to have forgotten the origin of quaternions, and from it that of the various systems of vectors. If one reads Hamilton carefully he will find that Hamilton was concerned with the extensions of number along the lines of algebra, starting with negative numbers and the so-called imaginary. In other words he was studying the theory of hypernumbers and made considerable progress along this line. These are not invariants, any more than other numbers are invariants, and their development antedates that of invariants by several years. The idea of linear transformations was superimposed upon them and is not the outcome of their study. It is only in a very superficial sense that one could consider that the idea of negative, or of $\sqrt{-1}$, was an outcome of natural number. These notions may be called with some justification an outcome of operations upon number, but operations upon number and number itself are very different entities. Further to represent $\sqrt{-1}$ as a linear substitution, as Peano does, is again only an isomorphism, and should not be confused with an identity. Keeping this in mind we will see that a system of vectors is a system of algebraic numbers in an extended sense, and all formulas are algebraic formulas, when algebra is thus understood. They do not primarily deal with multiplexes at all.* A quaternion is an algebraic number of the type of the ordinary complex number, and so long as we deal only with its powers and their combinations with positive and negative numbers, a quaternion differs in no sense from a complex number. Further it is an individual entity and not a set of four numbers. This Hamilton very well understood though he used sets of four positive or negative numbers in the various representations of quaternions. When this fact is clearly understood all the nonsense about geometric numbers and the like vanishes. The expressions in the formulas of such hypernumbers are not there because of certain invariancies, but because of the properties of these algebraic numbers. That they turn out to be invariants for orthogonal or other transformations of the so-called coefficients is due to such facts as that in any quaternion formula as $S\alpha\beta$, or Sqr, we have such relations as

$$S \cdot t\alpha t^{-1} t\beta t^{-1} = S \cdot \alpha\beta, \qquad S \cdot tqt^{-1} trt^{-1} = Sqr.$$

Therefore when Professor Study laments the fact that investigators have not tried to list the sets of "fundamental invariants" he might as well be concerned over the fact that not much study has been made

* The „multiplex" character means merely that from *one* equation between vectors arise an *infinity* of equations between ordinary numbers, all dependent on any N of the set which are linearly independent.

of the arithmetics that are developed in such hypernumber systems, though the latter line of development is now started by Dickson. Hence the charge that the theory of invariants has not existed for such investigators is not well-founded. And further in the application of such hypernumbers to geometric problems no concern need be felt over invariants, for where the expressions do not depend upon coordinate planes, they are ipso facto all invariant expressions. The reviewer has pointed this out elsewhere* with reference to the Einstein relativity theory, showing that when such theory is stated in terms of an N-dimensional vector calculus, there are no coordinates involved, hence a change of coordinate references has nothing to do with the problem. The thesis that all physical phenomena should be so represented that the same expressions will hold whatever the system of coordinates is actually realised by removing all systems of coordinates. Indeed the difficulties dragged in by coordinates have no business there at all. And to assert, as Professor Study does, that there is no geometry of a sound logical character without coordinates, is to ignore the whole of Euclid, or any other system of synthetic geometry. And to assert further that this is mixing up the phenomenal world, and questions of epistemology, with geometry, is to show that one has not yet clearly seen what is going on in the study of synthetic geometry. Space forbids entering into a full explanation, but we may point out that the mind studies many types of ideal or non-material "constructions", and among these we find the ones that enter geometry alongside of the "arithmetic", and one has the same kind of reality or validity as the other.

The reviewer is quite well aware of all the notions that have clung to that of vector calculus, and has always maintained that those systems that were mere shorthand or symbolisms for coordinate expressions were not worthy the name of vector calculus. Nothing can be accomplished by them which cannot also be done equally well without them, and in so far Professor Study is right in insisting that if one utilises the algebra of invariants he can get along just as well. There is of course some small gain in the use of the vectors but it is a matter of taste whether one prefers to write either of the two sets of expressions (the first is Study's, the second is quaternions)

(AP) or $m_1(\varphi)$, $(AP)\,(A'P)$ or $m_1(\varphi^2)$, $(AP')\,(A'P'')\,(A''P)$ or $m_1(\varphi^3)$, $\frac{1}{6}(AA'A'')\,(PP'P'')$ or $m_3(\varphi)$, (XU) or $S\rho\tau$, $(XA)\,(PU)$ or $S\rho\varphi\tau$, $(XA)\,(PA')\,(P'U)$ or $S\rho\varphi^2\tau$, $\frac{1}{2}(XPP')\,(AA'U)$ or $S\rho\psi(\varphi)\tau$, $(SX)^3$ or $S\rho\varphi\rho\varphi^2\rho$, $(\Sigma U)^3$ or $S\tau\ddot{\varphi}\tau\dot{\varphi}^2\tau$ etc.

One form is no more intelligible than the other, and expresses no more than the other so long as we are planting our feet on the artificial

* *General vector algebra*, TRANSACTIONS OF THIS SOCIETY, vol. 24 (1922), pp. 195–244.

ground of coordinates. But when we remember that the invariant symbols have no meaning for metrical work till the coordinate planes are given and the coordinates known, while the quaternion symbols are directly interpretable without any such artificial reference, the two sets come to have a quite different significance. There is at least as much advantage in the quaternion case as in the use of trigonometric solutions over these of rectangular coordinate geometry.

The essence of the matter is that when one is looking at every problem in geometry from the projective point of view, then projective invariants will be useful. When, however, he is looking at geometry from the metric point of view, then the direct metric analysis is far more advantageous. To use the artificial relations Study has on pages 35 et seq. is to be forgetful of the fact that mathematics is not so poverty-struck as to possess but one garment.

In order to be absolutely clear and definite, a word will not be amiss as to the significance of expressions to a quaternionist (whatever they may mean to others who merely dabble in vector analysis). The expression

$S\alpha\beta$ means the lengths of α and β multiplied by the cosine of their exterior angle;

$V\alpha\beta$ means the vector perpendicular positively to the plane α, β, whose length is the product of their lengths by the sine of the exterior angle;

$S\alpha\beta\gamma$ is the volume of the parallelepiped whose edges are α, β, γ;

$\varphi\rho$ is a linear vector function of ρ, that is, is such that we have $\varphi(x\alpha + y\beta) = x\varphi\alpha + y\varphi\beta$.

In other words *$S\alpha\beta$ does not mean $-ax-by-cz$, $S\alpha\beta\gamma$ does not mean* a three-rowed determinant, etc. This is what vector analysts like Burali-Forti and Marcolongo and their predecessors mean by "*autonomous expressions*", in the sense of not being related in any way to coordinate systems. Apparently some men have not yet seized this simple idea.

As far as the work before us has a constructive purpose,—that of developing a systematic text which shall be an introduction or even a manual of the theory of invariants under the linear homogeneous group or some of its subgroups, using the notion of triplex as a foundation,—we find a successful accomplishment of the purpose. The author has shown, what other writers on invariants have not, that invariants can be used, together with the symbolic notation peculiar to invariants, for the study of any geometric problems expressible by ternary forms. In a succeeding volume he expects to continue into n-ary forms. Professor Study has memoirs of long standing on these subjects, and has produced a very satisfactory book for these uses. A detailed list of contents is not necessary; the treatment is quite complete. If any book can resurrect the theory of invariants this one will.

J. B. SHAW

SHORTER NOTICES

Cours d'Analyse Infinitésimale. By Ch. J. de la Vallée Poussin. Vol. I, 5th ed. Paris, Gauthier-Villars, 1923.

An earlier edition of this Cours was reviewed in this BULLETIN of 1915. This latest edition has been very largely recast and the first volume only deals with the classic topics treated in the standard French *Cours*. The matter printed in large type in the older editions constitutes the bulk of this volume and only Riemann integrals are considered. The material referring to point-set theory, measurable functions and Lebesgue integrals which was printed in small type in the old edition will find place in a subsequent volume.

This arrangement seems much better than the old one for it will permit of a homogeneous development more in harmony with the historical evolution of the ideas involved.

Even in following the conventional order of the French treatises, de la Vallée-Poussin displays his usual elegance and simplicity of presentation so that the most hackneyed matters acquire a new interest. Thus on page 39 will be found a neat and simple proof that the familiar functional equations

$$f(x+y) = f(x) + f(y), \qquad \Phi(x+y) = \Phi(x)\,\Phi(y)$$

have unique solutions of the form $f(x) = ax$ and $\Phi(x) = A^x$ provided f and Φ are *borné* in an arbitrarily small interval (O, e).

The treatment of indeterminate forms is the best the reviewer knows of and extremes of functions of one or more variables are carefully handled.

Chapter III on derivatives and differentials of functions of several variables is extremely well done and the conditions of Young and Schwarz for the inversion of the order of derivation are neatly and briefly established.

Even the usually arid subject of formal integration gains with his deft treatment an elegance and beauty rarely found in text books on the calculus. Multiple integrals, change of variable, the theorems of Green and Stokes are carefully treated and conventions of sign, so often loosely passed over, are fully gone into. Attention is called to the clever evaluation of the error in Simpson's rule.

The book ends with a chapter on series where the usual tests including Kummer's are set forth. In conclusion it may be said that this is one of the most valuable handbooks on modern analysis in any language and an English translation of it would be a welcome addition to our literature of the subject.

M. B. PORTER

Lezioni sulla Teoria dei Numeri Algebrici. By Luigi Bianchi, Bologna, Nicola Zanichelli, 1923, 640 pp.

After an examination of the contents of this book the reviewer feels that it serves a double purpose. On account of the very comprehensive treatment of the subject the book is an excellent work for reference. Moreover, on account of the manner in which the author coordinates the various notions used in the development, it will undoubtedly prove an excellent work for any one who wishes to gain a clear insight into the theory of algebraic numbers.

A brief résumé of the contents of the various chapters will serve to show the ground covered and also to give an idea of how the various topics are correlated in the development.

Chapter one deals with the elements of the theory of matrices and their applications to systems of linear equations and systems of linear forms. It also contains a brief study of forms relative to a modular system $(f_1, f_2, \ldots, f_n)$ where the f_i are linear forms. This leads to the proof of Minkowski's well known theorem regarding linear forms. Some space is devoted to the geometric interpretation.

In the second chapter the author develops the elements of the theory of numbers for the Gaussian field $k(i)$, $(i^2 = -1)$. This development contains the proofs of the fundamental theorems whose generalizations are contained in the succeeding chapters. The chapter concludes with the consideration of further examples of quadratic number fields, showing how the law of unique factorization, which was seen to hold for $k(i)$, breaks down.

The third chapter contains the development of the algebraic side of the general theory of algebraic number fields.

Chapter four is devoted to the theory of units in an algebraic number field. The author shows the existence of units, and of fundamental systems of units.

The fifth chapter contains the development of the theory of ideals and chapter six is devoted to congruences with respect to an ideal modulus with special consideration of prime ideal moduli. In this connection the author treats the subject of quadratic residues which he has already treated, in Chapter 2, for the field $k(i)$. Quadratic residues in a quadratic field are given special consideration.

Chapter seven contains the theory of equivalence of ideals and the separation of ideals into classes. The group of classes and its invariants are studied and a proof of the finiteness of the number of classes is given. This chapter also contains a consideration of the correspondence between ideals and decomposable forms and the relations between classes of ideals and classes of forms, and the multiplication of ideals and composition of forms.

Chapter eight contains the proofs of the general theorems regarding

the factorization of the rational primes in an algebraic number field and its relation to the decomposition of $f(x)$ with respect to the modulus p, where $f(x) = 0$ is an equation one of whose roots generates the field. The last part of the chapter considers the prime factors of the field discriminant and also the factorization of the rational primes in cyclotomic fields.

The first part of chapter nine contains the theory of orders (Dedekind, Ordnung; Hilbert, Ring) in a field, and the ideals of any order. The second part of the chapter is devoted to Galois domains and their sub-domains.

Chapter ten deals with the analytic theory of algebraic numbers and develops the transcendental expression for the number of classes of ideals in a field.

Chapter eleven contains the development of the expression for the number of classes of the quadratic and cyclotomic fields and the proof of Dirichlet's theorem regarding arithmetic progressions.

G. E. WAHLIN

Report on Radiation and the Quantum-Theory. By J. H. Jeans. London, Fleetway Press, Ltd., 1924. 86 pp.

The first edition of this report was written in 1914 and was published by the Physical Society of London. In preparing the new edition the writer has omitted some of the parts in which he says there is an *apologia* for the defects and inconsistencies of the theory and has made numerous additions which add to the charm and lucidity of the presentation. The author's remark "that the quantum theory need no longer be considered on the defensive" aptly describes the present situation; in fact some writers have a feeling that the apologia must now be made for the classical electrodynamics.

In the first chapter Jeans points out that the smallness of the total density of radiant energy in temperature-equilibrium with matter compared with that of the heat-energy in the matter cannot be explained on the basis of Newtonian mechanics or by a supposed analogy with a vibrating elastic system immersed in a fluid such as air or water, for it is known by experience that the energy of the elastic system is finally transformed into heat energy of the fluid. The relative smallness of the density of the radiation at 0° C is quite startling in the case cited, being 4×10^{-5} ergs per cubic centimeter as compared with 8×10^{9} ergs per cubic centimeter in the matter.

In the next chapter it is shown how the classical electrodynamics in combination with the kinetic theory of gases leads to the radiation formula which is associated with the names of Rayleigh and Jeans.

This formula is correct in the region of long wave-lengths or at very high temperatures when the density of radiation is very large

and one of the guiding principles of quantum theory in its later developments is that a formula of quantum theory should reduce to that given by the classical theory under the conditions just mentioned.

Jeans then gives an account of the development of the quantum-theory, describing, in particular, the derivation of Planck's formula which was given by Einstein and the investigations of Poincaré and Fowler regarding the nature of the laws of motion that can lead to Planck's formula.

Chapters IV and V are devoted to Bohr's theory of spectra and to Einstein's theory of the photoelectric effect while Chapter VI contains an account of Debye's theory of the specific heat of solids. Much additional matter is contained in Chapter VII which deals with the dynamics of the quantum theory. The articles on conditionally periodic systems and Bohr's principle of correspondence will be found useful as an introduction to the more complete works mentioned in the text.

The report ends with a short discussion of the physical basis for the quantum theory in which the views expressed are practically the same as in the first edition. The idea of light-quanta is regarded with disfavor but it is perhaps unfortunate that the new report should have been completed just when a big development of the theory of light-quanta was taking place. The beautiful discoveries of Compton and Duane have shown that the phenomena of the scattering and diffraction of X-rays are not incompatible with the idea of light-quanta and the development of the idea of light-molecules by Wolfke, Bothe and de Broglie has shown that Planck's law of radiation is indeed obtained by slight modifications of arguments which previously led to Wien's law. Thus Pauli's remarkable application of the Compton-Debye theory of scattering to the equilibrium between free electrons and cavity radiation has been completed in this way by Bothe and Schrödinger and it now appears that the distribution of energy among the different wave-lengths may be unaltered by collisions of various types, including even the collisions between waves of light and sound which were considered first by Brillouin.

H. Bateman

Bibliographie des Déterminants à plus de Deux Dimensions. By Maurice Lecat. Louvain, 1924. 16 pp.

Those interested in determinants will welcome this short bibliography by an author who has made the subject peculiarly his own. It consists of three parts; a bibliography with 121 entries arranged alphabetically, an interesting chronological table, and a list of minor contributions to the subject. It is worthy of note that 50 out of the titles listed are by M. Lecat himself.

J. H. M. Wedderburn

Handy Methods of Geodetic Astronomy for Land Surveyors. By T. G. Gribble. London, Potter, 1921. 27 pp.

The particular feature of this pamphlet is to show how, "by means of some rapidly made diagrams, the stars available at any time and place can be at once identified, and their observation facilitated". Two sets of diagrams are used: I. Diagrams for a fixed latitude, II. Diagrams for all latitudes.

Under I, the variable laid off horizontally is the declination of the celestial body in question, its altitude, azimuth or hour angle being laid off vertically. Curves are shown giving for objects of various declinations (*a*) the altitude when on the meridian, (*b*) the altitude and hour angle when in azimuth 90° E or W, (*c*) the altitude and azimuth when in hour angle 90°, (*d*) the altitude, azimuth and hour angle of a circumpolar star at elongation.

Under II, the variable laid off horizontally is again the declination, and two sets of curves, for various latitudes at intervals of 5°, are drawn giving for objects of different declinations (*a*) the altitude and hour angle of objects due east or due west, (*b*) the altitude and azimuth of objects whose hour angle is six hours, (*c*) the altitude, azimuth and hour angle of circumpolar stars at elongation.

The curves are sketched in the usual way through points whose positions have been computed by the usual formulas of spherical trigonometry. It is suggested that a 20-inch slide rule be used for this computation and that the diagrams be drawn on a large scale. The method has the advantages and disadvantages always associated with graphical methods. The general relations involved are clearly shown, but it is doubtful whether sufficiently accurate values could be obtained in this way, and the labor involved in the construction of the diagrams would be considerable.

C. H. Currier

Differential Equations in Applied Chemistry. By F. L. Hitchcock and C. S. Robinson. New York, Wiley and Sons, 1923. vi+110 pp.

The purpose of this little text is to introduce the student in chemistry to the processes of building up differential equations where he has to use them, and to carry out the integrations necessary for the complete solution. It is full of examples from the subject and will be interesting for this reason to the student of mathematics as well. The actual equations solved belong to a very limited number of types but are of the kind a chemist meets. The book fills a gap in this part of applied mathematics. Its importance to the physical chemist depends upon what the latter has as his goal.

J. B. Shaw

The Alpha, Beta, Gamma Navigation Tables. By H. B. Goodwin. London, Potter, 1921. 4 + 54 pp.

The Alpha and Beta Tables, arranged in parallel columns, give respectively the natural cosines and logarithmic cosines of angles at intervals of 1′ from 0° to 90°. They are intended for use in cases where formulas of the type $\cos z = \cos a \pm \cos b$ where a and b are given and $\log \cos z$ is required for further computation. To facilitate the addition or subtraction of the natural cosines of a and b, each natural cosine in the tables has been multiplied by 1,000, and the resulting product it regarded as so many seconds of arc. Thus $\cos 60° = \cdot 500$, which is regarded as 500 seconds of arc, and is given in the table as 8′ 20″. It is claimed that it is easier to add or subtract quantities thus expressed in minutes and seconds of arc than to add or subtract four-place decimals, and this claim seems reasonable. After the natural cosines have been combined in this way, the parallel column arrangement makes it possible to get at once the logarithmic cosine of z without the trouble of taking out the angle itself and looking up its logarithmic cosine in a separate table.

The Gamma Tables give the value of the logarithm of 1/vers h (where h is the hour angle) to four decimal places for values of h between 0 hours and 24 hours at intervals of 5 seconds. It is claimed that these tables simplify the solution of the problem of fixing a ship's position either by the Marcq system or by the older Sumner process. More generally they facilitate the solution of problems which involve the two cases of spherical trigonometry:

(1) Given three sides of a triangle to find an angle.

(2) Given two sides and the included angle to find the third side.

In both these cases the necessary formulas can be reduced to forms involving the sums and differences of cosines, and 1/vers h.

C. H. Currier

Zentral-Perspektive, by Hans Freyberger. Revised by J. Vonderlinn. Sammlung Göschen, No. 57, 2d edition, Berlin and Leipzig, Walter de Gruyter & Co. 1923. 148 pp.

This little book originally written by an architect, emphasises the practical and artistic aspect of the subject and may very well serve as a first introduction to this fascinating chapter of applied geometry. The figures are well drawn and the exercises well selected, so that the reader who wishes to acquire a working knowledge of perspective can readily accomplish this by reconstructing the figures on a larger scale in connection with a comprehensive study of the text.

Arnold Emch

NOTES

The second number of volume 26 of the TRANSACTIONS OF THIS SOCIETY (April, 1924) contains the following papers: *On the expansion of analytic functions in series of polynomials,* by J. L. Walsh; *Operations with respect to which the elements of a boolean algebra form a group,* by B. A. Bernstein; *Isometric W-surfaces,* by W. C. Graustein; *Space time continua of perfect fluids in general relativity,* by L. P. Eisenhart; *Equivalent rational substitutions,* by J. F. Ritt; *Extension of Bernstein's theorem to Sturm-Liouville sums,* by E. Carlson; *An existence theorem,* by E. Hille; *On the complete independence of the postulates for betweenness,* by W. E. Van de Walle; *A new set of postulates for betweenness, with proof of complete independence,* by E. V. Huntington.

The third number of volume 46 of the AMERICAN JOURNAL OF MATHEMATICS contains: *Geometric aspects of the abelian modular functions of genus four,* by A. B. Coble; *The curve of ambience,* by F. Morley; *On a class of invariant subgroups of the conformal and projective groups in function space,* by I. A. Barnett.

The second number of volume 25, series 2, of the ANNALS OF MATHEMATICS contains: *A contribution to the theory of closed chains,* by A. Arwin; *On the characterization of the set of points of λ-continuity,* by H. Blumberg; *On the Ampère-Cauchy derived functions,* by H. L. Smith; *Note on certain seminvariants of n-lines,* by L. P. Copeland; *The Fredholm theory of Stieltjes integral equations,* by C. A. Fischer; *Rectilinear congruences referred to special surfaces,* by M. C. Foster; *Parallel propagation of a vector around an infinitesimal circuit in an affine manifold,* by J. L. Synge.

In 1918 the Mathematical Society of Charkow found it impossible to continue the publication of its COMMUNICATIONS. After an interval of more than five years a new journal has now been founded, under the auspices of this Society and edited by Professor S. Bernstein; it is called SECTION MATHÉMATIQUE DES ANNALES SCIENTIFIQUES DES INSTITUTIONS SAVANTES DE L'UKRAINE.

Another new mathematical journal, ARCHIMEDE, RASSEGNA BIMESTRALE DI MATEMATICA, FISICA E INGEGNERIA, has recently been founded at Palermo, as the organ of the Società Universitaria di Scienze Matematiche. It is edited by G. Priulla.

At the meeting of the British Association for the Advancement of Science at Toronto in August, 1924, Professor Horace Lamb was elected president, for the meeting to be held in Southampton in 1925. Sir W. H. Bragg, as president of the section of mathematical and physical sciences, delivered an address on *Crystal structure.* On the

occasion of this meeting, the University of Toronto conferred the honorary degree of doctor of science on Sir Ernest Rutherford, retiring president of the Association.

Among the officers of the National Research Council for 1924–25 are the following: Dr. G. E. Hale, honorary chairman; Professor A. A. Michelson, first vice-chairman; Professor R. A. Millikan, third vice-chairman and chairman of the division of foreign relations; Professor J. S. Ames, chairman of the division of physical sciences.

Professor J. C. McLennan, of the University of Toronto, has been elected president of the Royal Society of Canada.

On the occasion of his seventy-fifth birthday, Professor Felix Klein was elected honorary member of the Deutsche Mathematiker-Vereinigung and of the Gesellschaft für angewandte Mathematik und Mechanik, and received the honorary degree of doctor rerum politicarum from the University of Berlin.

Professor R. A. Millikan, of the California Institute of Technology, has received the honorary degree of doctor of science from Trinity College, Dublin.

Professor A. S. Eddington, of Cambridge University, has been elected an honorary member of the American Astronomical Society.

Professor M. I. Pupin, of Columbia University, has received the honorary degree of doctor of science from Princeton University.

The private scientific library of the late Professor A. G. Webster, of Clark University, has been purchased by the Riverbank Laboratories, Geneva, Illinois, and is now housed there as a separate collection.

The two Benjamin Peirce Instructorships in Mathematics at Harvard University (see this BULLETIN, vol. 21, p. 315) are again open for general competition. Applications for the year 1925–26, accompanied by the necessary papers, should reach Professor Birkhoff not later than February 15, 1925, and all inquiries relating to these appointments should be addressed to him.

Professors M. Amaldi, of the University of Padua, and L. Silla, of the University of Genoa, have been transferred to the Higher School of Architecture, recently established at Rome.

Associate Professor R. Wavre has been promoted to a full professorship at the University of Geneva.

The following have been admitted as privat-docents: D. Gigli, at the University of Pavia; A. Hammerstein, at the University of Berlin.

Dr. B. B. Baker, of the University of Edinburgh, has been appointed professor of mathematics at Royal Holloway College, University of London.

Dr. S. Brodetsky, of the University of Leeds, has been promoted to the chair of applied mathematics.

Professor Sydney Chapman, of the University of Manchester, has been appointed successor to Professor A. N. Whitehead as chief professor of mathematics at the Imperial College of Science and Technology, South Kensington.

Dr. T. M. Cherry has been appointed associate professor of applied mathematics at the University of Manchester.

Mr. E. A. Milne, of the Solar Physics Laboratory at Cambridge, has been appointed to the Beyer chair of applied mathematics at the University of Manchester.

Miss Katherine S. Arnold, for the past three years professor of mathematics at Constantinople Woman's College, has been appointed a member of the administrative staff of the American Association of University Women, effective September, 1924.

Mr. E. F. Baxter has been appointed assistant lecturer in mathematics at the University of Sheffield.

At Bryn Mawr College, Professor Charlotte A. Scott has retired; she is succeeded as head of the department of mathematics by Professor Anna J. Pell. Dr. D. V. Widder has been appointed associate in mathematics.

Mrs. Ethelwyn R. Beckwith has been appointed professor of mathematics at the College for Women, Western Reserve University, for 1924-25, to serve during the sabbatical leave of Professor Anna H. Palmié.

Dr. H. E. Bray, of Rice Institute, has been promoted to an assistant professorship of mathematics.

Miss Evelyn T. Carroll, of Wells College, has been promoted to an assistant professorship of mathematics.

Professor E. C. Coker, of Winthrop College, has been appointed professor of mathematics and astronomy at the University of South Carolina.

Assistant Professor A. R. Crathorne, of the University of Illinois, has been promoted to an associate professorship of mathematics.

Professor Ira M. De Long, who retires this year, on a Carnegie pension, as Professor and Head of the Department of Mathematics at the University of Colorado, has been elected President of the Mercantile Bank and Trust Company of Boulder, Colorado.

Miss Leslie Gaylord, of Agnes Scott College, has been promoted to an assistant professorship of mathematics.

Assistant Professor Lachlan Gilchrist, of the University of Toronto, has been promoted to an associate professorship of physics.

Assistant Professor W. H. Hill, of the State Teachers' College of Pittsburg, Kansas, has been promoted to an associate professorship of mathematics.

Professor J. M. Howie, of Alma College, has been appointed head of the department of mathematics at Nebraska Wesleyan University.

Dr. J. H. Jeans, of Cambridge University, has been appointed research associate at Mount Wilson Observatory.

Associate Professor A. S. Merrill, of the University of Montana, has been promoted to a full professorship of mathematics.

Miss Georgia E. Robinson, of the University of Missouri, has been appointed acting head of the department of mathematics at Shorter College, Rome, Georgia.

Mr. J. C. Tinner has been appointed professor of mathematics at Wilberforce University.

Mr. W. M. Whyburn, of the South Park Junior College, Beaumont, Texas, has been appointed assistant professor of mathematics at Texas Agricultural and Mechanical College.

The following appointments to instructorships are announced:
Amherst College, Mr. C. S. Porter;
Brooklyn Polytechnic Institute, Mr. D. H. MacPherson;
Bryn Mawr College, Miss Anna M. Lehr;
Coker College, Miss Nellie C. Stokes;
Colorado School of Mines, Mr. E. P. Martinson;
Cornell University, Mr. H. A. Hoover;
University of Iowa (astronomy) Dr. D. H. Menzel;
Hood College, Miss Evelyn P. Wiggin;
Lehigh University, Mr. C. A. Balof;
University of Oklahoma, Mr. A. E. Anderson;
Purdue University, Mr. J. C. Bennett;
University of Wisconsin, Mr. H. S. Pollard.

Professor August Gutzmer, of the University of Halle, died May 10, 1924, at the age of sixty-four. He had been on the editorial staff of the Jahresbericht der Deutschen Mathematiker-Vereinigung since 1894, and was sole editor from 1901 to 1921.

Lucien de la Rive, the Swiss mathematical physicist, died at Geneva May 4, 1924, at the age of ninety years.

Professor T. Kotof, of the University of Charkow, died in 1923, at the age of twenty-eight years.

Professor R. H. Jude, of Rutherford College, Newcastle-on-Tyne, died June 1, 1924, at the age of seventy.

Mr. Charles Leudesdorf, registrar of Oxford University and author of mathematical papers, died August 10, 1924, at the age of seventy-one.

Professor J. G. Longbottom, of the department of mechanics at the Royal Technical College, Glasgow, died June 6, 1924, at the age of fifty-four.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

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